Chapter 04

Convolution Product

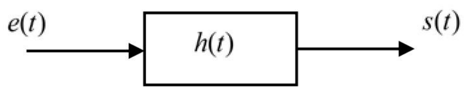
4.1 Introduction

The convolution product is a tool of great importance in physics. We encounter it, for example, whenever we study:

- The transmission of a signal by a device,
- An electrical impulse as a function of time,
- An image represented by a function of one or two variables,
- In diffraction.

4.2 Definition

Either $e(t)$ And $h(t)$ two functions can be integrated on $\Re$. We call the convolution product of $e(t)$ by $h(t)$ the function noted $e(t) * h(t)$, defined by the assumed convergent integral.



$$s(t) = e(t) * h(t) = \int_{-\infty}^{+\infty} e(\tau)\, h(t-\tau) d\tau$$

In the special case where the functions $e(t)$ And $h(t)$ are causal on $\Re -$ we obtain:

$$s(t) = e(t) * h(t) = \int_{0}^{t} e(\tau)\, h(t-\tau) d\tau$$

4.3 Properties of the convolution product

1) Commutativity

$$e(t) * h(t) = h(t) * e(t)$$

$$h(t) * e(t) = \int_{-\infty}^{+\infty} h(\tau)\, e(t-\tau) d\tau$$

2) Associativity

$$s_1(t)*[s_2(t)*s_3(t)]=[s_1(t)*s_2(t)]*s_3(t)$$

3) Distributivity

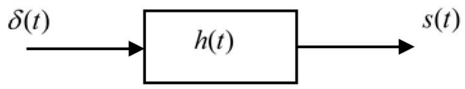
$$s_1(t)*[s_2(t)+s_3(t)]=[s_1(t)*s_2(t)]+[s_1(t)*s_3(t)]$$

4) Derivation of the convolution

$$\frac{d}{dt}[s_1(t)*s_2(t)]=s_1(t)*\frac{ds_2(t)}{dt}+\frac{ds_1(t)}{dt}*s_2(t)$$

5) Neutral element of the convolution

The neutral element of the convolution product is the Dirac distribution.



$$s(t)=h(t)*\delta(t)=\int_{-\infty}^{+\infty} h(\tau)\,\delta(t-\tau)d\tau=h(t)$$

Convolution by a Dirac peak therefore returns the signal in its entirety.

It follows that convolving a signal by a shifted Dirac distribution allows the signal to be translated.

Indeed:

$e(t)$ and $h(t)*\delta(t-a)=h(t-a)$

4.4 Graphical interpretation of convolution

The calculation of the convolution therefore consists of calculating the surface of the product $e(t)$ And $e(\tau)h(t-\tau)$.

The signal $h(t-\tau)$ is simply the initial signal $h(\tau)$, returned in time to give $h(\tau)$, then translated from t.

Then calculating all the surfaces obtained by "sliding" $h(t-\tau)$, that is, for all offsets of t, we obtain the convolution product for all t.

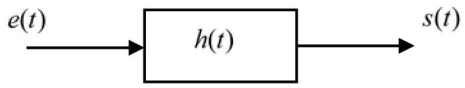
<u>Noticed</u>:

To calculate a convolution product $e(t)*h(t)$:

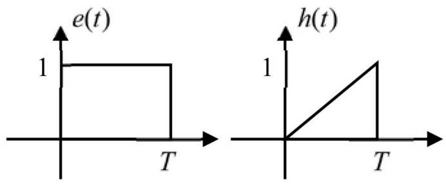
1. The variable t is replaced by τ.
2. The first signal must be kept.
3. Find the symmetric of the second with respect to the ordinate axis $h(\tau)\rightarrow h(-\tau)$.
4. Shift this signal by time t, $h(-\tau)\rightarrow h(t-\tau)$.
5. Multiply the two signals obtained $e(\tau)h(t-\tau)$, and finally integrate the result.

Example 4.1:

Calculate the output: $s(t) = e(t) * h(t) = \int_{-\infty}^{+\infty} e(\tau)\, h(t-\tau) d\tau$



With : $e(t) = \begin{cases} 1 & if\ 0 \leq t \leq T \\ 0 & if\ not \end{cases}$ And $h(t) = \begin{cases} \frac{1}{T}t & if\ 0 \leq t \leq T \\ 0 & if\ not \end{cases}$



4.5 Convolution Theorem

The Fourier transform has the remarkable property of exchanging the convolution and multiplication in the ordinary sense of functions. Let us now calculate the Fourier transform of the convolution product of two functions, that is to say to calculate the TF of $s(t) = e(t) * h(t)$.

$$S(f) = \int_{-\infty}^{+\infty} s(t) e^{-j2\pi ft} dt$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e(\tau) h(t-\tau) d\tau . e^{-j2\pi ft} dt$$

$$S(f) = \int_{-\infty}^{+\infty} e(\tau) d\tau \int_{-\infty}^{+\infty} \left[h(t-\tau) . e^{-j2\pi ft} dt \right]$$

We use the time translation property:

$$H(f) e^{-j2\pi f\tau} = \int_{-\infty}^{+\infty} \left[h(t-\tau) . e^{-j2\pi ft} dt \right]$$

$$S(f) = H(f) \int_{-\infty}^{+\infty} e(\tau) . e^{-j2\pi f\tau} d\tau$$

We have:

$$E(f) = \int_{-\infty}^{+\infty} e(\tau) . e^{-j2\pi f\tau} d\tau$$

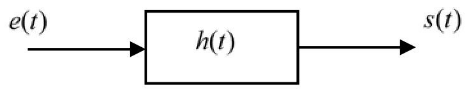
Eventually:

$$S(f) = E(f) . H(f)$$

$$\begin{cases} S(f) = E(f).H(f) \\ S(p) = E(p).H(p) \end{cases} \Leftrightarrow s(t) = e(t) * h(t)$$

Example 4.2:

Consider the following system:



With : $e(t) = h(t) = p_T(t)$

1- Calculate the output $s(t)$ by two methods:

1.1 Direct calculation with graphical interpretations (**homework**).

1.2 Using the convolution theorem.

2- Graphically represent the signal $s(t)$.