Chapter 03

Laplace transform

3.1 Introduction

The Laplace transform is a transformation that allows us to move from the time domain to the Laplace domain and allows the resolution in the frequency domain of problems posed in the time domain.

3.2 Definition

To any function f of variable t, such as $f(t)$ For $t<0$, we match a function F of complex variable p (with $p=\tau+jw$).

τ : is the real part, w :is the imaginary part.

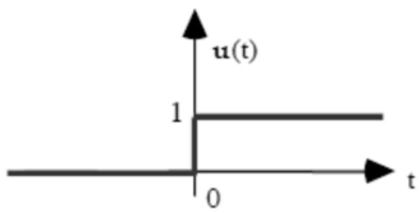
We define the Laplace transform:

$$F(p)=L[f(t)]=\int_{0}^{+\infty} f(t)\, e^{-pt} dt$$

3.2.1 LT of the step function

It is a signal defined by:

$$u(t)=\begin{cases}0 & if\ t<0\\ 1 & if\ t\geq 0\end{cases}$$



We now calculate $U(p)$ the Laplace transform of $u(t)$.

$$U(p)=\int_{0}^{+\infty} u(t)\, e^{-pt} dt$$

$$=\int_{0}^{+\infty} 1\, e^{-pt} dt$$

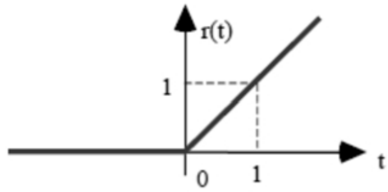
$$=-\frac{1}{p}\left[e^{-pt}\right]_{0}^{+\infty}=\frac{1}{p}$$

$$U(p)=L[u(t)]=\frac{1}{p}$$

3.2.2 LT of the ramp function

It is defined by:

$$r(t) = \begin{cases} 0 & if\ t < 0 \\ t & if\ t \geq 0 \end{cases}$$



We now calculate $R(p)$ the Laplace transform of $r(t)$.

$$R(p) = \int_0^{+\infty} r(t)\, e^{-pt} dt$$

$$= \int_0^{+\infty} t\, e^{-pt} dt$$

$$= \left[-\frac{t}{p} e^{-pt} \right]_0^{+\infty} - \int_0^{+\infty} -\frac{1}{p} e^{-pt} dt$$

$$= 0 + \frac{1}{p} \int_0^{+\infty} e^{-pt} dt \quad \Rightarrow R(p) = TL[r(t)] = \frac{1}{p^2}$$

3.3 Properties and theorems

$f(t)\ (t \geq 0)$	$F(p) = L\{f(t)\}$
$f(t)$	$\int_0^{+\infty} e^{-pt} f(t) dt$
$\lambda_1 f_1(t) + \lambda_2 f_2(t)$	$\lambda_1 F_1(p) + \lambda_2 F_2(p)$
$\frac{df(t)}{dt}$	$pF(p) - f(0)$
$\frac{d^n f(t)}{dt^n}$	$p^n F(p) - \sum_{r=n+1}^{r=2n} p^{2n-r} . \frac{d^{(r-n-1)} f(0)}{dt^{(r-n-1)}}$ With: $f^{(r-n-1)}(0) = \left. \frac{d^{(r-n-1)} f(t)}{dt^{(r-n-1)}} \right\|_{t=0}$ **Example 3.1:** $L\left\{ \frac{d^3}{dt^3} f(t) = p^3 F(p) - p^2 f(0) - pf'(0) - f''(0) \right\}$
$\int_0^t f(t)$	$\frac{1}{p} L\{f(t)\}$
$\int_0^t \int_0^t ... \int_0^t f(t).dt^n$; (With conditions zero initials)	$\frac{F(p)}{p^n}$

$f(kt)$	$\frac{1}{k}.F\left(\frac{p}{k}\right)$
$f\left(\frac{t}{k}\right)$	$k.F(k.p)$
$e^{-a.t}f(t)$	$F(p+a)$
$f(t-\tau)\ ;\ t \geq \tau$	$e^{-p.\tau}F(p)$
$t.f(t)$	$\frac{d}{dp}F(p)$
$f(0^+) = \lim_{p\to\infty}\{pF(p)\} \quad and \quad f(\infty) = \lim_{p\to 0}\{pF(p)\}$	

3.4 Laplace transform of usual functions

The function	Laplace transform
$\delta(t)$	1
$u(t)$	$\frac{1}{p}$
$t.u(t)$	$\frac{1}{p^2}$
$t^n.u(t)$	$\frac{n!}{p^{n+1}}$
$e^{-a.t}.u(t)$	$\frac{1}{p+a}$
$t.e^{-a.t}.u(t)$	$\frac{n!}{(p+a)^{n+1}}$
$\sin(\omega_0 t).u(t)$	$\frac{\omega_0}{p^2+\omega_0^2}$
$\cos(\omega_0 t).u(t)$	$\frac{p}{p^2+\omega_0^2}$
$e^{-a.t}\sin(\omega_0 t).u(t)$	$\frac{\omega_0}{(p+a)^2+\omega_0^2}$
$e^{-a.t}\cos(\omega_0 t).u(t)$	$\frac{p+a}{(p+a)^2+\omega_0^2}$

3.5 Inverse Laplace Transform

The inverse transform can be expressed using Fourier integrals. If $F(p)$ is the Laplace Transform of a function $f(t)$, we have:

$$f(t) = TL^{-1}[F(p)] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(p)e^{pt}dp$$

Or c :is a constant, called the abscissa of convergence.

- This method only applies if the roots of the polynomial in the denominator are known, in other words, if the denominator is factorizable:

$$F(p) = \frac{N(p)}{D(p)} = \frac{\prod_{j=0}^{k}(p - z_k)}{\prod_{i=0}^{n}(p - p_i)} = k\frac{(p+z_1)(p+z_2)...(p+z_m)}{(p+p_1)(p+p_2)...(p+p_n)}$$

3.5.1 F(p) contains only distinct poles

$$F(p) = \frac{N(p)}{D(p)} = \frac{a_1}{p+p_1} + \frac{a_1}{p+p_1} + ... + \frac{a_n}{p+p_n}$$

With:

$$a_i = \left[\frac{N(p)}{D(p)}(p+p_i)\right]_{p=p_i}$$

a_i : constant called "residue at the pole $p = p_i$ ".

Example 3.1:

Find the Inverse Transform of $F(p) = \dfrac{p+3}{(p+1)(p+2)}$

Example 3.2:

Either $F(p) = \dfrac{p^3 + 5p^2 + 9p + 7}{(P+1)(p+2)}$

3.5.2 F(p) contains multiple poles

Either p_1 the multiple pole of $F(p)$, r being the multiplicity index of this pole.

$$F(p) = \frac{N(p)}{D(p)} \qquad D(p) = (p+p_1)^r (p+p_{r+1})(p+p_{r+2})...(p+p_n)$$

SO $F(p)$ is written:

$$F(p) = \frac{b_r}{(p+p_1)^r} + \frac{b_{r-1}}{(p+p_1)^{r-1}} + ... + \frac{b_1}{(p+p_1)} + \frac{b_r}{(p+p_1)^r} + \frac{a_{r+1}}{(p+p_{r+1})} + \frac{a_{r+2}}{(p+p_{r+2})} + ... + \frac{a_n}{(p+p_n)}$$

With : $a_k = \left[\frac{N(p)}{D(p)} (p + p_k) \right]_{p=-p_k} \quad (k = r+1, r+2, ..., n)$

And:

$$b_r = \frac{1}{0!} \left[\frac{N(p)}{D(p)} (p + p_1)^r \right]_{p=-p_1}$$

$$b_{r-1} = \frac{1}{1!} \left[\frac{d}{dp} \left\{ \frac{N(p)}{D(p)} (p + p_1)^r \right\} \right]_{p=-p_1}$$

$$b_{r-j} = \frac{1}{\mathrm{j}!} \left[\frac{d^j}{dp^j} \left\{ \frac{N(p)}{D(p)} (p + p_1)^r \right\} \right]_{p=-p_1}$$

$$b_1 = \frac{1}{(r-1)!} \left[\frac{d^{r-1}}{dp^{r-1}} \left\{ \frac{N(p)}{D(p)} (p + p_1)^r \right\} \right]_{p=-p_1}$$

<u>Noticed</u>: $TL^{-1} \left[\frac{1}{(p + p_1)^n} \right] = \frac{t^{n-1}}{(n-1)!} e^{-p_1 t}$

Example 3.3:

Either: $F(p) = \frac{p^2 + 2p + 3}{(p+1)^3} \Rightarrow F(p) = \frac{b_3}{(p+1)^3} + \frac{b_2}{(p+1)^2} + \frac{b_1}{(p+1)}$

3.5.3 F(p) contains complex conjugate poles

Let p_1 And p_2 the 2 complex poles conjugate, then:

$$F(p) = \frac{N(p)}{D(p)} = \frac{\alpha_1 p + \alpha_2}{(p + p_1)(p + p_2)} + \frac{a_3}{p + p_3} + ... + \frac{a_n}{p + p_n}$$

With:

α_1 And α_2 residues at the poles p_1 And p_2 : $(\alpha_1 p + \alpha_2)_{p=-p_1} = \left[\frac{N(p)}{D(p)} (p + p_1)(p + p_2) \right]_{p=-p_1 \ or \ p=-p_2}$

Example 3.4:

Find the Inverse Transform of $F(p) = \frac{p+1}{p(p^2 + p + 1)}$

3.6 Solving differential equations by the symbolic method

We will consider the solution of the following equation:

$$\begin{cases} a\dfrac{d^2y(t)}{dt^2}+b\dfrac{dy(t)}{dt}+cy(t)=f(t) \\ C.I: y'(0)=A=cte^{\circ}, y(0)=B \end{cases}$$

We use the Laplace transform:

$$TL\left\{a\frac{d^2y(t)}{dt^2}+b\frac{dy(t)}{dt}+cy(t)\right\}=TL\{f(t)\}$$

$$a\left(p^2Y(p)-py(0)-y'(0)\right)+b\left(pY(p)-y(0)\right)+cY(p)=F(p)$$

$$Y(p)\left(a.p^2+b.p+c\right)=F(p)+a.B.p+A.a+b.B \Rightarrow Y(p)=\frac{F(p)+B(a.p+b)+a.A}{a.p^2+b.p+c}$$

So the solution to the differential equation is the inverse Laplace transform of:

$$y(t)=TL^{-1}\{Y(p)\}=TL^{-1}\left\{\frac{F(p)+B(a.p+b)+a.A}{a.p^2+b.p+c}\right\}$$

Example 3.5:

$$\begin{cases} \dfrac{d^2y(t)}{dt^2}+y(t)=0 \\ C.I: y'(0)=0, y(0)=1 \end{cases}$$