Chapter 02

Fourier analysis

2.1 Introduction

There are 2 areas of signal representation:

- Temporal representation: $s(t)$ with t : time.
- Frequency representation: $S(f)$ with f : the frequency.

Difficulties in the time Domain → Frequency domain.

2.2 Fourier Series

2.2.1 Trigonometric Fourier Series

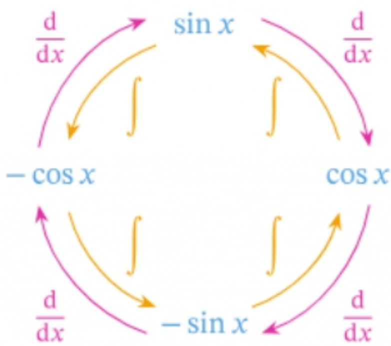
For periodic signals of period $T_0 = \frac{1}{f_0}$ it is given by:

$$s_f(t) = a_0 + \sum_{n=1}^{+\infty}[a_n.\cos(2\pi n f_0 t) + b_n.\sin(2\pi n f_0 t)]\,; \qquad w_0 = 2\pi n f_0 t\,.$$

$a_0 = \frac{1}{T_0}\int_{T_0} s(t).dt$ → *the average value*=0, if $s(t)$ is odd.

$a_n = \frac{2}{T_0}\int_{T_0} s(t).\cos(2\pi n f_0 t).dt$ =0, if $s(t)$ is odd.

$b_n = \frac{2}{T_0}\int_{T_0} s(t).\sin(2\pi n f_0 t).dt$ =0, if $s(t)$ is even.



Example 2.1:

Let the function be: $s(t) = t$ for $-\pi \le t \le \pi$.

1- Plot this function in the frequency domain (amplitude=1).

2- Determine the Trigonometric Fourier series associated with the periodic function ($T_0 = 2\pi$).

2.2.2 Alternative Fourier Series

It is defined by: $s_f(t) = A_0 + \sum_{n=1}^{+\infty} A_n.\cos(2\pi n f_0 t - \varphi_n)$

With : $A_0 = a_0;\quad A_n = \sqrt{a_n + b_n};\quad \varphi_n = \arctan\frac{b_n}{a_n}$

Example 2.2:

1- Determine the Alternating Fourier series of the function in Example 1.

2.2.3 Complex Fourier Series

It is defined by: $s_f(t) = \sum_{n=-\infty}^{+\infty} c_n.e^{+j2\pi n f_0 t}$;

with : $c_n = \frac{1}{T_0}\int_{T_0} s(t).e^{-j2\pi n f_0 t}dt$; And $c_0 = \frac{1}{T_0}\int_{T_0} s(t)\,dt$

We have: $e^{j\theta} = \cos\theta + \sin\theta$; And $e^{-j\theta} = \cos\theta - \sin\theta$.

We replace the exponential form in the integral of c_n :

So:

$$c_n = \frac{1}{T_0}[\int_{T_0} s(t).\cos(2\pi n f_0).dt - j\int_{T_0} s(t).\sin(2\pi n f_0).dt]$$

Then:

$c_n = \frac{a_n}{2} - j\frac{b_n}{2}$; for $n = 0$ $c_0 = \frac{1}{T_0}\int_{T_0} s(t).dt = a_0$. $\rightarrow$ (The average value)

We therefore conclude the relationship with the Trigonometric Fourier series:

$$\begin{cases} a_n = 2.\mathrm{Re}\{c_n\} \\ b_n = -2.\mathrm{Im}\{c_n\} \\ a_0 = c_0 \end{cases}$$

Example 2.3:

1- Determine the Complex Fourier series of Example 2.1.

2- Deduce the Trigonometric Fourier series.

3- Represent the amplitude and phase spectrum of the Complex form.

Exercise 2.1 (homework):

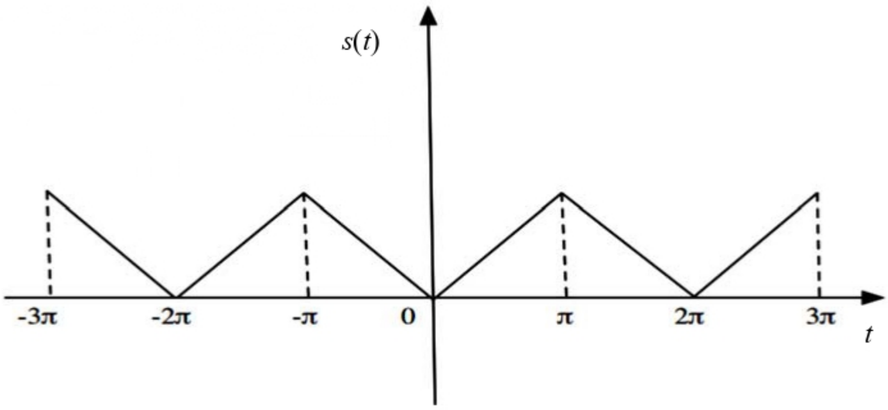
1- Find the mathematical expression of the function $s(t)$ (amplitude=2) on the interval $[-\pi\ ;+\pi]$.

2- Determine the Trigonometric Fourier series associated with the periodic function ($T_0 = 2\pi$).

3- Deduce the Alternative Fourier series.

4- Determine the Complex Fourier series.

5- Calculate and graph the amplitude and phase spectrum of the complex form.



2.3 Fourier Transform

2.3.1 Introduction

- Observing a signal in the time domain is often insufficient to perform signal analysis. → Analysis in the frequency domain.
- In the case of continuous-time periodic signals, the Fourier series decomposition allows us to deduce a frequency representation.
- How to obtain the frequency representation of any deterministic (non-periodic) signals?

New mathematical tool: → the Fourier Transform

Fourier transform Let a continuous-time signal be $s(t)$ *(non-periodic)*

$$S(f) = F\{s(t)\} \int_{-\infty}^{+\infty} s(t)e^{-j2\pi ft}dt$$

– $S(f)$ is the FT of $s(t)$ or even the specter of $s(t)$.

– The continuous variable f represents the frequency in Hz.

Inverse Fourier Transform

It defines by:

$$s(t) = F^{-1}\{S(f)\} = e^{+j2\pi ft}df$$

$S(f)$ is, in general, a complex-valued function. The FT of a signal can be written in exponential form:

$$S(f) = |S(f)|e^{j\varphi(f)}$$

Amplitude spectrum $\left|S(f)\right| = \sqrt{\mathrm{Re}(S(f)^2 + \mathrm{Im}(S(f)^2}$

Phase spectrum $\varphi(f) = \arctan\left[S(f)\right] = \arctan\left(\dfrac{\mathrm{Im}(S(f)}{\mathrm{Re}(S(f)}\right)$

- $\left|S(f)\right|$ And $\varphi(f)$ are functions of the continuous variable f ,we speak of continuous spectra (in opposition to Fourier series).
- If $s(t)$ is real-valued, so $\left|S(f)\right|$ is even and $\varphi(f)$ is odd.

Example 2.4: Consider the following function:

$s(t) = e^{-\alpha t}.u(t); \quad \alpha > 0$

1- Graph this function

2- Find the Fourier Transform of this function.

3- Calculate and represent the amplitude and phase spectrum.

2.3.2 Properties of the Fourier transform

1) Linearity

$F\left\{a.x(t) + b.y(t)\right\} = a.X(f) + b.Y(f)$

2) Symmetry

$$\begin{aligned} if \quad & s(t) \xrightarrow{F} S(f) \\ & S(t) \xrightarrow{F} s(f) = s(-f) \end{aligned}$$

Example 2.5:

be the signal $x(t) = p_T(t)$

1- Graph the function $x(t)$.

2- Find the Fourier Transform of $x(t)$.

3- Use the symmetry property to calculate the FT $s(t) = \sin c(2\pi f_0 t)$.

3) FT of the Dirac impulse

$$F\left\{\delta(t)\right\} = \int_{-\infty}^{+\infty} \delta(t) e^{-j2\pi f t} dt$$

We use the Dirac momentum property:

$$\int_{-\infty}^{+\infty} \delta(t) x(t).dt = x(0)$$

$$\int_{-\infty}^{+\infty} \delta(t) x(t - t_0).dt = x(t_0)$$

$$F\{\delta(t)\} = e^0 = 1$$

$$\delta(t) \xrightarrow{F} 1$$

What is the Fourier transform of 1 ($F\{1\} = ?$)

$$F\{1\} = \int_{-\infty}^{+\infty} 1.e^{-j2\pi ft} dt$$

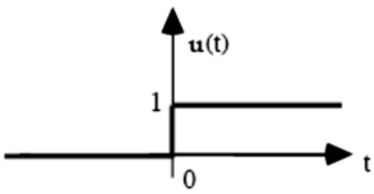
$$= [\frac{e^{-j2\pi ft}}{j2\pi f}]_{-\infty}^{+\infty} = ?$$

We use the property of symmetry:

$$\delta(t) \xrightarrow{F} 1$$

$$1 \xrightarrow{F} \delta(f)$$

4) FT of the step function and the sign function



$$u(t) = \frac{1}{2} + \frac{1}{2}\text{sgn}(t)$$

$$\text{sgn}(t) = \lim_{\alpha \to 0} \text{s}(t)$$

$$\text{s}(t) = \begin{cases} e^{-\alpha t} & if\ t > 0 \\ -e^{\alpha t} & if\ t < 0 \end{cases} \quad \alpha \to 0$$

$$F\{\text{sgn}(t)\} = \lim_{\alpha \to 0} TF\{s(t)\}$$

$$F\{s(t)\} = \int_{-\infty}^{0} -e^{\alpha t} e^{-j2\pi ft} dt + \int_{0}^{+\infty} e^{-\alpha t} e^{-j2\pi ft} dt$$

$$= -\frac{1}{\alpha - j2\pi f} + \frac{1}{\alpha + j2\pi f}$$

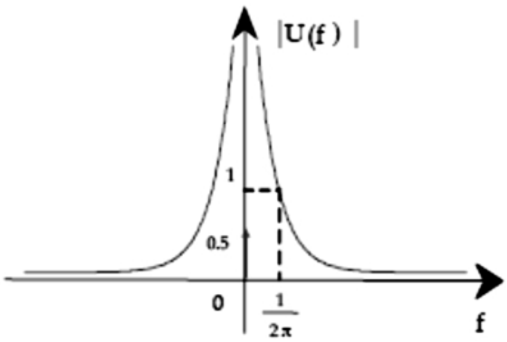
$$= \frac{-j4\pi f}{\alpha^2 + 4\pi^2 f^2}$$

$$F\{\text{sgn}(t)\} = \lim_{\alpha \to 0} TF\{s(t)\}$$

$$= \frac{-j}{\pi f}$$

$$F\{\text{sgn}(t)\} = \frac{1}{j\pi f}$$

$$F\{u(t)\} = F\left\{\frac{1}{2} + \frac{1}{2}\mathrm{sgn}(t)\right\}$$
$$= \frac{1}{2j\pi f} + \frac{1}{2}\delta(f)$$



5) FT of a periodic signal

If $s(t)$ is a periodic signal of period T_0 :

The complex Fourier series is:

$$s_f(t) = \sum_{n=-\infty}^{+\infty} c_n.e^{+j2\pi nf_0t} \text{ ; with : } c_n = \frac{1}{T_0}\int_{T_0} s(t).e^{-j2\pi nf_0t}dt$$

So:

$$F\{s(t)\} = \int_{-\infty}^{+\infty} s_f(t).e^{-j2\pi ft}dt$$
$$= \int_{-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} c_n.e^{+j2\pi nf_0t}.e^{-j2\pi ft}dt$$
$$= \sum_{n-\infty}^{+\infty} c_n \int_{-\infty}^{+\infty} 1.e^{-j2\pi t(f-nf_0)}dt$$

We have:

$$\int_{-\infty}^{+\infty} 1.e^{-j2\pi ft}dt = \delta(f) \text{ , SO } \int_{-\infty}^{+\infty} 1.e^{-j2\pi t(f-nf_0)}dt = \delta(f-nf_0)$$

Finally, the FT of a periodic signal:

$$F\{s(t)\} = \sum_{n-\infty}^{+\infty} c_n\delta(f-nf_0)$$

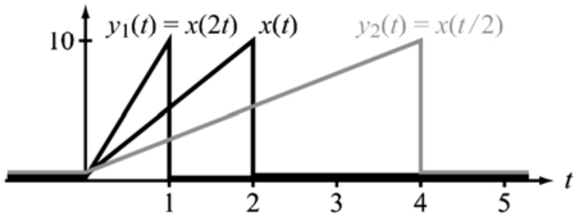
<u>Noticed :</u>in can calculate c_n by this formula: $c_n = \frac{1}{T_0}.F\left\{s_{T_0}(t)\right\}\Big|_{f=nf_0}$

Example 2.6:

1- Find the Fourier transform of the signal $x(t)$ from example 2.1:

6) Homothety or scale factor ($a > 0$ ***)***

$$s(t) \xrightarrow{F} S(f)$$
$$s(a.t) \xrightarrow{F} \frac{1}{|a|} S(\frac{f}{a})$$



Demonstration: We ask:

$$u = a.t \Rightarrow t = \frac{1}{a} u$$
$$\Rightarrow dt = \frac{1}{a} du$$

So:

$$S(f) = \int_{-\infty}^{+\infty} \frac{1}{a} s(u).e^{-j2\pi \frac{f}{a} u} du$$
$$= \begin{cases} \frac{1}{a} S(\frac{f}{a}) & if \quad a > 0 \\ -\frac{1}{a} S(\frac{f}{a}) & if \quad a < 0 \end{cases}$$

7) Translation in time

$$s(t) \xrightarrow{F} S(f)$$
$$s(t+t_0) \xrightarrow{F} S(f)e^{+j2\pi f t_0}$$
$$s(t-t_0) \xrightarrow{F} S(f)e^{-j2\pi f t_0}$$

Example 2.7:

be the signal $s(t) = \delta(t-1)$.

1- Graph the signal $s(t)$.

2- Use the time translation property to calculate the FT of $s(t)$.

3- Calculate and graph the amplitude and phase spectra of the signal $S(f) = F\{s(t)\}$ depending on f.

8) Frequency translation

$$s(t) \xrightarrow{F} S(f)$$
$$s(t)e^{-j2\pi f_0 t} \xrightarrow{F} S(f+f_0)$$
$$s(t)e^{+j2\pi f_0 t} \xrightarrow{F} S(f-f_0)$$

Example 2.8:

1- Find the FT of the following signal: $s(t) = \cos(2\pi f_0 t)$.

2- Deduce the TF from:

2-1. $y(t)=\sin(2\pi f_0 t)$

2-2. $z(t)=p_h(t)e^{+j\pi f_0 t}$

9) Derivation in time

$$s(t)\xrightarrow{F}S(f)$$
$$\frac{ds(t)}{dt}\xrightarrow{F}j2\pi f.S(f)$$
$$\vdots$$
$$\frac{d^n s(t)}{dt^n}\xrightarrow{F}\left(j2\pi f\right)^n.S(f)$$

Example 2.9:

Either the following signal: $s(t)=tri_T(t)$.

1- Graphically represent the signal $s(t)$ and its derivative $\frac{ds(t)}{dt}$.

2- Deduce the mathematical expression of the signal $\frac{ds(t)}{dt}$.

3- Find the signal FT $s(t)$ using the derivation property.

10) The time integral

$$s(t)\xrightarrow{TF}S(f)$$
$$\int s(t)\xrightarrow{TF}\frac{1}{j2\pi f}.S(f)$$
$$\vdots$$
$$\int\cdots\int s(t)\xrightarrow{TF}\frac{1}{\left(j2\pi f\right)^n}.S(f)$$

Example 2.10: Consider the following signal: $s(t)=\delta(t+\frac{T}{2})-\delta(t-\frac{T}{2})$.

1- Graphically represent the signal $s(t)$ and its integral $\int s(t)$.

2- Deduce the mathematical expression of the signal $\int s(t)$.

3- Find the signal FT $s(t)$ using the integral property.

11) Frequency derivation

$$s(t)\xrightarrow{F}S(f)$$
$$\left(-j2\pi t\right).s(t)\xrightarrow{F}\frac{dS(f)}{df}$$
$$\vdots$$
$$\left(-j2\pi t\right)^n.s(t)\xrightarrow{F}\frac{d^n S(f)}{df^n}$$

Example 2.11:

Consider the following signal: $s(t) = t.e^{-\alpha t}u(t)$.

1- Find the signal FT $s(t)$ using the frequency derivation property.

12) Modulation

$$s(t) \xrightarrow{F} S(f)$$

$$s(t)\cos(2\pi f_0 t) \xrightarrow{F} \frac{1}{2}\left[S(f+f_0)+S(f-f_0)\right]$$

2.4 Perceval's Theorem

The energy of a signal $s(t)$ East :

$$E = \int_{-\infty}^{+\infty} \|s(t)\|^2 \, dt$$

$$= \int_{-\infty}^{+\infty} s(t).s^*(t)dt$$

$$= \int_{-\infty}^{+\infty} s^*(t) \int_{-\infty}^{+\infty} S(f)e^{+j2\pi ft} df dt$$

$$E = \int_{-\infty}^{+\infty} S(f)df \int_{-\infty}^{+\infty} s^*(t)e^{+j2\pi ft} dt$$

We have:

$$S^*(f) = \int_{-\infty}^{+\infty} s^*(t)e^{+j2\pi ft} dt$$

So:

$$E = \int_{-\infty}^{+\infty} S^*(f)S(f)df$$

$$= \int_{-\infty}^{+\infty} \|S(f)\|^2 df$$

Eventually:

$$E_{s(t)} = E_{S(f)} \Rightarrow \int_{-\infty}^{+\infty} \|s(t)\|^2 \, dt = \int_{-\infty}^{+\infty} \|S(f)\|^2 \, df$$

Example 2.12:

Use either Perceval's theorem to calculate the energy of the following signal:

$s(t) = A.T \sin c(2\pi f_0 t)$.

2.5 Table of spectrum