Tutorial N.o6

Exercise 1 Given the three points $(0, 1), (1, 0.5), and\ (3, 0.25)$ for the function $f(x)$:

1. Determine the Lagrange polynomial that passes through these points.
2. Find an approximation of $f(1.5)$.
3. Knowing that $f(x) = 1 - (x + 1)$, calculate the maximum error when approximating $f(x)$ with the Lagrange polynomial $P(x)$, and compare it with the exact error.

Exercise 2:

Given the polynomials $P(x)\ and\ Q(x)$ defined as follows:

0	0	1	2
$P(x)$	-6	3	21
$Q(x)$	10	15	40

Find the points of intersection using the Lagrange interpolation method.

Exercise 3: Using the following data, find an approximation of $f(4.5)$ with a Newton polynomial of degree 2:

x	1.0	3.0	5.0	7.0
$f(x)$	0.0000	1.2528	1.6094	1.9456

Exercise 4:

1. Using Newton's interpolation method, calculate an approximation of $\sqrt{1.6}$. Take x0=1 x_1 =1, x_2 =2, and x_3=3.
2. Calculate the error committed

Exercise 5 (Supplementary):

Determine the Lagrange polynomial for the function $f(x) =| x |$ that passes through the points with x-coordinates $-1 , -0.5 , 0 , 0.5 , and\ 1$ 1.

Solution of Tutorial N.o6

Exercise 1:

x_i	$x_0 = 0$	x_1=1	x_2=2
$f(x_i)= y_i$	y_0=1	$y_1 = 0.5$	y_2=0.25

<u>1. Lagrange Polynomial Passing Through the 3 Points:</u>

We have 3 points, so the degree of the polynomial is n ≤ 2.

$$P_2(x) = \sum_{k=0}^{2} \frac{L_K(x)}{L_K(x_K)} y_K = \frac{L_0(x)}{L_0(x_0)} y_0 + \frac{L_1(x)}{L_1(x_1)} y_1 + \frac{L_2(x)}{L_2(x_2)} y_2$$

For $K = 0$:

$$\frac{L_0(x)}{L_0(x_0)} = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(0-x_2)} = \frac{(x-1)(x-3)}{(0-1)(0-3)} = \frac{1}{3}(x-1)(x-3)$$

For $K = 1$:

$$\frac{L_1(x)}{L_1(x_1)} = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0)(x-3)}{(1-0)(1-3)} = -\frac{1}{2}x(x-3)$$

For $K = 2$:

$$\frac{L_2(x)}{L_2(x_2)} = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_2)} = \frac{(x-0)(x-1)}{(3-0)(3-1)} = \frac{1}{6}x(x-1)$$

Thus,

$P_2(x)$=$\frac{1}{3}(x-1)(x-3) - \frac{1}{2}x(x-3) + \frac{1}{6}x(x-1) = 0.125x^2 - 0.625x + 1.$

<u>2. Approximation of $f(1.5)$</u>

Using the Lagrange polynomial$P_2(x)$, we find:

$y(x) = P_2(1.5) = 0.125(1.5^2) - 0.625(1.5) + 1$

$y(1.5) \approx 0.344$

3. Calculation of the Maximum Error:

$$|f(x) - P_2(x)| \leq E_{Max}(x)$$

Where:

$$E_{Max}(x) = \frac{M}{(n+1)!}\prod_{i=0}^{2}|x - x_i|$$

$M = Max\{|f^{(3)}(\xi)|\}, \quad \xi \in [0,3]$

$f(x) = -\frac{1}{(x+1)^2}, \quad f'' = \frac{2}{(x+1)^3}, \quad\quad f''' = -\frac{6}{(x+1)^4}$

$M = Max\left\{\left|-\frac{6}{(\xi+1)^4}\right|\right\} = Max\left\{\left|\frac{6}{(\xi+1)^4}\right|\right\}, \quad \xi \in [0,3],$

$\forall\, x_1 \in [0,3], \quad\quad \forall\, x_2 \in [0,3],$

$x_1 < x_2 \Rightarrow x_1 + 1 < x_2 + 1 \Rightarrow (x_1+1)^4 < (x_2+1)^4 \Rightarrow \frac{1}{(x_1+1)^4} > \frac{1}{(x_2+1)^4} \Rightarrow \frac{6}{(x_1+1)^4} > \frac{6}{(x_2+1)^4}$

$\Rightarrow |f'''(x_1)| > |f'''(x_2)| \Rightarrow |f'''(x)| \searrow$

The maximum is obtained for $x = 0$. Thus,

$M = \frac{6}{(0+1)^4} = 6,$

$E_{Max}(x) = \frac{6}{3*2}|x||x-1||x-3|, \quad x \in [0,3]$

For $x = 1.5$, $E_{Max}(1.5) = |1.5||1.5-1||1.5-3|$

$E_{Max}(1.5) = 1.125$

$E_{exact} = |f(1.5) - P_2(1.5)||0.4 - 0.344|$

$E_{exact} = 0.056$

Comparison:

We observe that $E_{exact} < E_{Max}$.

Exercise 2:

Lagrange Polynomial

a. $\boldsymbol{P(x)}$

x_i	$x_0 = 0$	x_1=1	x_2=2
$f(x_i)$=y_i	y_0=-6	$y_1 = 3$	y_2=21

We have 3 points, so the degree of the polynomial is n ≤ 2.

$$P_2(x) = \sum_{k=0}^{2} \frac{L_K(x)}{L_K(x_K)} y_K = \frac{L_0(x)}{L_0(x_0)} y_0 + \frac{L_1(x)}{L_1(x_1)} y_1 + \frac{L_2(x)}{L_2(x_2)} y_2$$

For $K = 0$:

$$\frac{L_0(x)}{L_0(x_0)} = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-1)(x-2)}{(0-1)(0-2)} = \frac{1}{2}(x-1)(x-2)$$

For $K = 1$:

$$\frac{L_1(x)}{L_1(x_1)} = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0)(x-2)}{(1-0)(1-2)} = -x(x-2)$$

For $K = 2$:

$$\frac{L_2(x)}{L_2(x_2)} = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_2)} = \frac{(x-0)(x-1)}{(3-0)(3-1)} = \frac{1}{2}x(x-1)$$

Thus,

$P_2(x)$=$\frac{1}{2}(x-1)(x-2)(-6) - x(x-2)(3) + \frac{1}{2}x(x-1)(21)$

$P_2(x)$=$\frac{1}{2} = \frac{9}{2}x^2 + \frac{9}{2}x - 6.$

b. **Q(x)**

x_i	$x_0 = 0$	x_1=1	x_2=2
$f(x_i)$=y_i	y_0=10	$y_1 = 15$	y_2=40

We have 3 points, so the degree of the polynomial is n ≤ 2.

$$Q_2(x) = \sum_{k=0}^{2} \frac{L_K(x)}{L_K(x_K)} y_K = \frac{L_0(x)}{L_0(x_0)} y_0 + \frac{L_1(x)}{L_1(x_1)} y_1 + \frac{L_2(x)}{L_2(x_2)} y_2$$

For $K = 0$:

$$\frac{L_0(x)}{L_0(x_0)} = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(0-x_2)} = \frac{(x-1)(x-2)}{(0-1)(0-2)} = \frac{1}{2}(x-1)(x-2)$$

For $K = 1$:

$$\frac{L_1(x)}{L_1(x_1)} = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0)(x-2)}{(1-0)(1-2)} = -x(x-2)$$

For $K = 2$:

$$\frac{L_2(x)}{L_2(x_2)} = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_2)} = \frac{(x-0)(x-1)}{(3-0)(3-1)} = \frac{1}{2}x(x-1)$$

Thus,

$Q_2(x)$=$\frac{1}{2}(x-1)(x-2)(10) - x(x-2)(15) + \frac{1}{2}x(x-1)(40)$

$Q_2(x) = 10x^2 - 5x + 10.$

Exercise 3:

Approximation of $f(4.5)$:

Since the polynomial is of degree 2, we need only 3 points. We choose points with abscissas close to 4.5. These points are presented in the following table:

x_i	$x_0 = 3$	x_1=5	x_2=7
$f(x_i)$=y_i	y_0=1.2528	$y_1 = 1.6094$	y_2=1.9459

We have 3 points, so the degree of the polynomial is $n = 2$. The Newton polynomial passing through these 3 points is:

$P_2(x)$=$y_0 + \frac{\Delta y_0}{1!h^1}(x-x_0)$+$\frac{\Delta^2 y_0}{2!h^2}(x-x_0)(x-x_1)$

where: h = x_1 - x_0 = $x_2 - x_1$= 2

x_0 = 3 → y_0 = 1.2528

$$\Delta y_0 = y_1 - y_0 = 0.3566$$

x_1 = 5 → y_1 = 1.6094

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = -0.0201$$

$$\Delta y_1 = y_1 - y_0 = 0.3566$$

x_2 = 7 → y_2 = 1.9459

Thus:

$P_2(x)$=1.2528 + $\frac{0.3566}{2}$ $(x - 3)$ + $\frac{-0.0201}{2!2^2}$ $(x - 3)\,(x - 5)$

$P_2(x) = 1.2528 + 0.1783\,(x - 3) - 0.0025\,(x - 3)(x - 5)$

The approximation of $f(4.5)$ is:

$f(4.5) \approx P_2(4.5) = 1.2528 + 0.1783\,(4.5 - 3) - 0.0025\,(4.5 - 3)(4.5 - 5)$

$P_2(4.5)$ = 1.5221

Exercise 4:

1. Calculation of the Approximation of $\sqrt{\mathbf{1.6}}$

x_i	$x_0 = 1$	x_1=2	x_2=3
$f(x_i)$=y_i	y_0=1	$y_1 = 1.4142$	y_2=1.7320

The Newton polynomial passing through these 3 points is:

$P_2(x)$=$y_0 + \frac{\Delta y_0}{1!h^1}(x - x_0)$+$\frac{\Delta^2 y_0}{2!h^2}(x - x_0)(\,x - x_1)$

where: h = x_1 - x_0 = $x_2 - x_1$= 2

x_0 = 1 → y_0 = 1

$$\Delta y_0 = y_1 - y_0 = 0.4142$$

x_1 = 2 → y_1 = 1.4142

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = -0.0964$$

$$\Delta y_1 = y_1 - y_0 = 03178$$

x_2 = 3 → y_2 = 1.7320

Thus:

$P_2(x)$=1 + $\frac{0.4142}{1!1^1}$ $(x - 1)$ + $\frac{-0.0946}{2!1^2}$ $(x - 1)\,(x - 2)$

$P_2(x) = 1 + 0.4142\,(x - 1) - 0.0482\,(x - 1)(x - 2)$

The approximation of $f(\sqrt{\mathbf{1.6}})$ is:

$f(1.6) = = 1 + 0.4142\,(1.6 - 1) - 0.0482\,(1.6 - 1)(1.6 - 2)$

$P_2(1.6) \;= 1.2601$

2. Error Committed:

The error committed in approximating $\sqrt{\mathbf{1.6}}$ byP_2 (1.6) is:

$$E_{exact} = |f(1.6) - P_2(x)| = \left|\sqrt{1.6} - 1.2601\right| = 0.0048.$$

Exercise 5 (Supplementary):

x_i	$x_0 = -1$	x_1=-0.5	x_2=0	x_2=0.5	x_2=1
$f(x_i)$= y_i	y_0=1	$y_1 = 0.5$	y_2=0	y_2=0.5	y_2=1

1. Lagrange Interpolation Polynomial of the Function $\boldsymbol{f(x)}$ Passing Through the Above Points:

We have 5 points, so the degree of the polynomial is $n \leq 4$.

$$P_4(x) = \sum_{k=0}^{2} \frac{L_K(x)}{L_K(x_K)} y_K = \frac{L_0(x)}{L_0(x_0)} y_0 + \frac{L_1(x)}{L_1(x_1)} y_1 + \frac{L_2(x)}{L_2(x_2)} y_2 + \frac{L_3(x)}{L_3(x_3)} y_3 + \frac{L_4(x)}{L_4(x_4)} y_4$$

For $K = 0$:

$$\frac{L_0(x)}{L_0(x_0)} = \frac{(x - x_1)(x - x_2)(x - x_3)(x - x_4)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)(x_0 - x_4)}$$

$$= \frac{(x + 0.5)(x - 0)(x - 0)(x - 0)}{(-1 + 0.5)(-1 - 0)(-1 - 0.5)(-1 - 1)}$$

$$= \frac{2}{3}(x + 0.5)x(x - 0.5)(x - 1)$$

For $K = 1$:

$$\frac{L_1(x)}{L_1(x_1)} = \frac{(x - x_1)(x - x_2)(x - x_3)(x - x_4)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)(x_0 - x_4)}$$

$$= \frac{(x + 1)(x - 0)(x - 0.5)(x - 1)}{(-0.5 + 1)(0.5 - 0)(-0.5 - 0.5)(-0.5 - 1)}$$

$$= -\frac{8}{3}(x + 1)x(x - 0.5)(x - 1)$$

For $K = 2$:

$$\frac{L_2(x)}{L_2(x_2)} = \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)}$$

$$= \frac{(x+1)(x+0.5)(x-0.5)(x-1)}{(0+0.5)(0+0.5)(0-0.5)(0-1)}$$

$$= 4(x+1)(x+0.5)(x-0.5)(x-1)$$

For $K = 3$:

$$\frac{L_3(x)}{L_3(x_3)} = \frac{(x-x_1)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)}$$

$$= \frac{(x+1)(x+0.5)(x-0)(x-1)}{(0.5+1)(0.5+0.5)(-0.5-1)(0.5-1)}$$

$$= -\frac{8}{3}(x+1)(x+0.5)x(x-1)$$

For $K = 4$:

$$\frac{L_4(x)}{L_4(x_4)} = \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}$$

$$= \frac{(x+1)(x+0.5)(x-0)(x-0.5)}{(1+1)(1+0.5)(1-0)(1-0.5)}$$

$$= \frac{2}{3}(x+1)x(x+0.5)x(x-0.5)$$

Thus,

$P_4(x)$=$\frac{2}{3}(x+0.5)x(x-0.5)(x-1)(1) - \frac{8}{3}(x+1)x(x-0.5)(x-1)(0.5) + 4(x+1)(x+0.5)(x-0.5)(x-1)(0) - \frac{8}{3}(x+1)(x+0.5)x(x-1)(0.5)$+$\frac{2}{3}(x+1)x(x+0.5)x(x-0.5)(1)$

$P_4(x) = -\frac{4}{3}x^4 + \frac{7}{3}x^2$.