Tutorial No.5

<u>Exercise 1</u>: Consider the following differential equation:

$$\begin{cases} y' = y + x \\ y(0) = 1 \end{cases}$$

Compute the approximate solution of this equation at $x = 1$ using Euler's method, dividing the interval into 10 equal parts.

Knowing that the exact solution is $y_{exact} = -1 - x + e^x$, compare the result obtained with $y_{exact}(1)$.

<u>Exercise 2</u>: Consider the differential equation:

Consider the differential equation:

$$\begin{cases} y' = y + e^{2x} \\ y(0) = 2 \end{cases}$$

1. Using a step size h=0.1, perform one iteration of the Euler-Cauchy (modified Euler) method and calculate the error by comparing the result with the exact solution $y_{\text{exact}}(0.1)$
2. Using a step size h=0.05 h = 0.05 h=0.05, perform two iterations of the Euler-Cauchy method and calculate the error at $y(0.1)$ by comparing the result with the exact solution$y_{exact}(1)$).
3. Compute the ratio of the errors from parts 1) and 2), and provide a comment on the result obtained.

<u>Exercise 3</u>: Consider the following differential equation:

$$\begin{cases} y' = y + x + 1 \\ y(0) = 1 \end{cases}$$

1. Compute the approximate solution $y(0.2)$ using Euler's method and the 4th-order Runge-Kutta method with a step size $h = 0.1$.
2. For each method, calculate the error by comparing the result with the exact solution $y_{exact}(0.2)$1.0187370780. Comment on the results obtained.

Note: Use 9 significant digits after the decimal point.

Solution of Tutorial N.o5

Exercise 1

$$\begin{cases} y' = y + x \\ y(0) = 1 \end{cases}$$

1. Calculation of the approximate solution of the differential equation at $x = 1$ using Euler's method

The working interval is $[0, 1]$.

The step size h is: $\frac{1-0}{10} = 0.1$.

Moreover, we have: $f(x, y) = y + x$.

Thus, we can use Euler's method:

$$y(x_{n+1}) \approx y(x_n) + h\, f(x_n, y(x_n))$$

and successively obtain approximations of

$y(0.1), y(0.2), y(0.3), y(0.4), y(0.5), y(0.6), y(0.7), y(0.8), y(0.9),)\ and\ (\ y(1.0)$). The first iteration yields:

$$y(0.1) \approx y(0) + h\, f(0, 1) = 1 + 0.1(1 + 0) = 1.1.$$

Similarly, the second iteration gives:

$$y(0.2) \approx y(0.1) + h\, f(0.1, 1.1) = 1.1 + 0.1(1.1 + 0.1) = 1.22.$$

And the third iteration gives:

$$y(0.3) \approx y(0.2) + h\, f(0.2, 1.22) = 1.22 + 0.1(1.22 + 0.2) = 1.362.$$

The following table summarizes the results of the first ten iterations:

x_n	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7
$f(x_n)$	1.0000	1.1000	1.2200	1.3620	1.5282	1.72120	1.943122	2.1974342

0.8	0.9	1.0
2.48717762	2.815895382	3.1874849202

Thus, the approximate solution of the differential equation at $x = 1$ is:

$$y(1) = 3.1874849202$$

2. Comparison of the obtained result with the exact solution

The exact solution is: $y_{exact}(x) = 1 - x - e^x$.Thus, the exact solution at $x = 1$ gives:

$$y_{execte}(1) = 1 - 1 - e^1 = 0.2490787367$$

The error committed is:

$E_e = |\ y_{exact}(x) - y(1)| = |: y_{exact}(1) - y(1)| = |0.2490787367 - 3.1874849202|$=0.2490787367

Exercise 2

$$\begin{cases} y' = y + e^{2x} \\ y(0) = 2 \end{cases}$$

1. Calculation of the first iteration of the Euler-Cauchy method

The step size is $h = 0.1$.

Moreover, we have $f(x, y) = y + e^x$.

Thus, we can use the Euler-Cauchy method:

$$y(x_{n+1}) \approx y(x_n) + \frac{h}{2}\,[f(x_n, y(x_n) + f(x_{n+1}, y(x_{n1})]$$

with

$$y_{n1} = y(x_n) + h\,f(x_n, y(x_n))$$

and obtain the approximation of $y(0.1)$. The first iteration yields:

$$y_{n1} = y(0) + h\,f(0,2) = 2 + 0.1(2 + e^{2*0}\}) = 2.3$$

which is the result obtained using Euler's method. The second step gives:

$$y(0.1) = y(0) + \frac{h}{2}[\,f(0,2) + f(0.1{,}2{,}3)] = 2 + \frac{0.1}{2}[2 + e^{2*0} + 2.3 + e^{2*0.1}) = 2.326070138$$

Error committed

The exact solution is: $y_{execte}(x) = e^x + e^{2x}$.

The exact solution at 0.1 gives , $y_{execte}(0.1) = e^{0.1} + e^{2*0.1}$.=2.326573676, so the error comitted is

$E_{ec(h=0.1)} = |\ y_{execte}(0.1) - y(0.1)| = |2.326573676 - 2.326070138|$=0.000503538

2. Calculation of the first two iterations of the Euler-Cauchy method

The step size is $h = 0.05$.

Moreover, we have $f(x, y) = y + e^{2x}$.

Thus, we can use the Euler-Cauchy method and successively obtain the approximations of $y(0.05)$ and

$y(0.1)$. The first iteration yields:

$$y_{n1} = y(0) + h\,f(0.2) = 2 + 0.05(2 + e^{2*0}) = 2.15$$

which is the result obtained using Euler's method. The second step gives:

$$y_{n1=}y(0.05) + h\,[f(00.05{,}2.156379273) = 2.156379273 + 0.05(2.156379273 + e^{2*0.05})]]$$
$$= 2.319456782$$

Similarly, the second iteration gives:

$$y(0.05) \approx y(0) + \frac{h}{2}[f(0,2) + f(0.05, 2.15)] = 2 + \frac{0.05}{2}[2 + e^{2*0} + 2.15 + e^{2*0.05}]$$
$$= 2.156379273$$

The correction then leads to:

$$y(0.1) \approx y(0.05) + \frac{h}{2}[f(0.05{,}2.156379273) + f(0.1{,}2.319456782)]$$
$$= 2.156379273 + \frac{0.05}{2}[2.156379273 + e^{2*0.05} + 2.319456782 + e^{2*0.1}]$$
$$= 2.326439516$$

Error committed

The error committed is:

$E_{ec(h=0.1)} = |\, y_{exect}(0.1) - y(0.1)| = |2.326573676 - 2.326439516| = 0.00013416$

3. Error ratio

The error ratio is:

R = $\frac{E_{ec(h=0.1)}}{E_{ec(h=0.05)}}$=3.75≈ 2^2

It is clear that the difference between the exact solution and the approximate solution decreases by a factor of $3.75 \approx 2^2$ when the step size h is halved, which confirms that the Euler-Cauchy method is of order 2.

Exercise 3

1. Calculation of the approximation of $y(0.2)$:

The working interval is $[0, 0.2]$.

The step size h is: $h = 0.1$.

Moreover, we have: $f(x,y) = -y + x + 1$.

a. Euler's method

Thus, we can use Euler's method:

$$y(x_{n+1}) \approx y(x_n) + h\,f(x_n, y(x_n)),$$

and successively obtain approximations of $y(0.1)$ and $y(0.2)$. The first iteration yields:

$$y(0.1) \approx y(0) + h\,f(0,1) = 1 + 0.1(-1 + 0 + 1) = 1.$$

Similarly, the second iteration gives:

$$y(0.2) \approx y(0.1) + h\,f(0.1,1) = 1 + 0.1(-1 + 0.1 + 1) = 1.01.$$

Thus, the approximation of $y(0.2)$ using Euler's method is $y(0.2) \approx 1.01$.

b. Runge-Kutta method of order 4

Thus, we can use the Runge-Kutta method of order 4:

$k_1 = h\,f(x_n, y(x_n))$

$k_2 = h\,f(x_n + \frac{h}{2}, y(x_n) + \frac{k_1}{2})$

$k_3 = h\,f(x_n + \frac{h}{2}, y(x_n) + \frac{k_2}{2})$

$k_4 = h\,f(x_n + h, y(x_n) + k_3)$

$$y(x_{n+1}) \approx y(x_n) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

and successively obtain approximations of $y(0.1)$ and $y(0.2)$. The first iteration yields:

$k_1 = h\,f(0.1) = 0.1(-1 + 0 + 1) = 0$

$k_2 = h\,f\left(0 + \frac{0.1}{2}, 1 + \frac{0}{2}\right) = 0.1f(0.05{,}1) = 0.1(-1 + 0.05 + 1) = 0.005$

$k_3 = h\,f\left(0 + \frac{0.1}{2}, 1 + \frac{0}{2}\right) = 0.1f(0.05{,}1.0025) = 0.1(-1.0025 + 0.05 + 1) = 0.00475$

$k_4 = h\,f(0 + 0.1{,}1 + 0.00475) = 0.1f(0.1{,}1.00475) = 0.1(-1.00475 + 0.1 + 1)$

which leads to:

$$y(0.1) \approx y(0) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}(0 + 2 * 0.005 + 2 * 0.00475 + 0.009525) = 1.0048375$$

Similarly, the second iteration gives:

$k_1 = h\,f(0.1, 1.0048375) = 0.1(-1.0048375 + 0.1 + 1) = 0.00951625$

$k_2 = h\,f(0.1 + \frac{0.1}{2}, 1.0048375 + \frac{0.00951625}{2}) = 0.1\,f(0.15, 1.009595625)$

$\qquad = 0.1(-1.009595625 + 0.15 + 1) = 0.014040437$

$k_3 = h\,f(0.1 + \frac{0.1}{2}, 1.0048375 + \frac{0.014040437}{2}) = 0.1\,f(0.15, 1.011857718)$

$\qquad = 0.1(-1.011857718 + 0.15 + 1) = 0.013814228$

$k_4 = h\,f(0.1 + 0.1, 1.0048375 + 0.013814228) = 0.1\,f(0.2, 1.018651728)$

$\qquad = 0.1(-1.018651728 + 0.2 + 1) = 0.018134827$

which leads to:

$$y(0.2) \approx y(0.1) + \frac{1}{6}(\,k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \; 1.0048375 \; + \; \frac{1}{6}(0.00951625 \; + \; 2 \, * \, 0.014040437 \; + \; 2 \, * \, 0.013814228 \; + \; 0.018134827)$$

$$= \; 1.018730901$$

Thus, the approximation of $y(0.2)$) using the Runge-Kutta method of order 4 is $y(0.2) \approx$1.01873090.

2. Error committed

The exact solution is: $y_{exact}(0.2) \; = \; 1.018730780$

a. Error committed by Euler's method:

$E_e = |\, y_{exect}(0.2) - y(0.2)| = |1.018730780 \; - \; 1.01| = \; 0.00870750 = 0.873078 * 10^{-2}$

b. Error committed by the Runge-Kutta method of order 4:

E_{RK4}= $|\, y_{exect}(0.2) - y(0.2)|$ = |1.018730780 - 1.018730901| = 0.000000121 = $0.121 * 10^{-6}$

It is clear that Euler's method is much less accurate than the Runge-Kutta method of order 4. Therefore, it is preferable to use methods of as high an order as possible.