Tutorial No4.

<u>Exercise 1</u>: Consider the following integral:

$$J = \int_0^2 \sqrt{x}\ dx$$

1. Calculate the integral using the trapezoidal method.
2. Calculate the integral using Simpson's method.
3. Compare the two results obtained with the exact value.

Exercise 2:

Consider the following integral:

$$J = \int_0^{\pi} sinx^2\ dx$$

1. Calculate the integral using the generalized trapezoidal method with 5, then 10 intervals.
2. Knowing that the exact value is 0.7726, compare the results obtained with the exact value.

Exercise 3:

Consider the following integral:

$$J = \int_0^1 \frac{1}{x+1} dx$$

1. Calculate the integral using the generalized Simpson's method with 4, then 8 intervals.
2. Compare the results obtained with the exact value.
3. Calculate the maximum error committed in the two previous cases.

Exercise 4

Consider the function $f(x)$ defined by the following table:

x_i	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$
$f(x_i)$	0	0.382683	0.707107	0.923880	1

1. Calculate the integral $\int_0^{\frac{\pi}{2}} f(x)$ using the generalized trapezoidal method.
2. Recalculate this integral using the generalized Simpson's method.
3. Knowing that$f(x) = sinx$ compare the results obtained with the exact value.
4. Find the number of intervals nn necessary to achieve an error of 10^{-6}using the generalized Simpson's method.

Exercise 5 (Supplementary): Consider the following integral:

$$J = \int_0^1 e^x\, dx$$

1. Calculate the integral using the generalized trapezoidal method with 4, then 8 intervals.
2. Compare the results obtained with the exact value.
3. Calculate the maximum error committed in the two previous cases.

Solution of TD n° 4

Exercise 1:

1. Calculation of the integral using the trapezoidal method

$\int_a^b f(x)dx \cong J_T =$ (b - a)$\left[\frac{1}{2}f(a) + \frac{1}{2}f(b)\right]$

$\int_a^b \sqrt{x}\,dx \cong J_T =$ (2 - 0)$\left[\frac{1}{2}\sqrt{x} + \frac{1}{2}\sqrt{2}\right] \approx 1.4142$

2. Calculation of the integral using the Simpson method

$\int_a^b f(x)dx \cong J_S =$ (b - a)$\left[\frac{1}{6}f(a) + \frac{1}{6}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f(b)\right]$

$\int_a^b \sqrt{x}\,dx \cong J_S =$ (2 - 0)$\left[\frac{1}{6}\sqrt{0} + \frac{4}{6}\sqrt{1} + \frac{1}{6}\sqrt{2}\right] \approx 1.8047$

3. Comparison

$J_{exacte} \approx \int_0^2 \sqrt{x}\,dx \cong J_T = \int_0^2 \sqrt{3}\,dx \cong \left[\frac{(x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^2 = \frac{2}{3}\left[\sqrt{x^3}\right]_0^2 = \frac{2}{3}\left[\sqrt{x^3}\right]_0^2 \approx 1.8856$

$|J_{exacte} - J_T|$=$|1.8856 - 1.4142|$=0.4714

$|J_{exacte} - J_S|$=$|1.8856 - 1.0809|$=0.0809

We therefore conclude that the approximation of the integral given by the Simpson method is better than that obtained by the trapezoidal method.

Exercise 2

1. Calculation of the integral using the generalized trapezoidal method

a. With 5 small intervals

$$J = \int_0^\pi sinx^2\,dx$$

$$h = \frac{b-a}{n} = \frac{\pi - 0}{5} = \frac{\pi}{5}$$

x_i	x_0=$\pi/5$	x_1=$\pi/5$	x_2=$2\pi/5$	x_3=$3\pi/5$	x_4=$4\pi/5$	x_5=π
$f(x_i)$	0.0000	0.3846	1.0000	-.03999	0.0333	-0.4303

With:

$\int_a^b f(x)dx \cong J_{T_{G_5}} = \frac{h}{2}\left[f(x_0) + f(x_5) + 2\big(f(x_1) + f(x_2) + f(x_3) + f(x_4)\big)\right]$

$J_{T_{G_5}} = \frac{\pi}{10}[0.0000 - 0.4303 + 2(0.3846 + 1.0000 - 0.3999 + 0.03333)] = 0.5044$

b. With 10 small intervals :

$$h = \frac{b-a}{n} = \frac{\pi - 0}{10} = \frac{\pi}{10}$$

x_i	x_0=$\pi/5$	x_1=$\pi/10$	x_2=$\pi/5$	x_3=$3\pi/10$	x_4=$2\pi/5$	x_5=$\pi/2$
$f(x_i)$	0.0000	0.0958	0.3846	0.7760	1.0000	0.6243

x_6=$3\pi/5$	x_7=$7\pi/10$	x_8=$4\pi/5$	x_9=$9\pi/10$	x_{10}=π
-.03999	-0.9924	0.0333	0.9902	-.0.4303

With:

$\int_a^b f(x)dx \cong J_{T_{G_{10}}} = \frac{h}{2}\left[f(x_0) + f(x_{10}) + 2\big(f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + \cdots + f(x_9)\big)\right]$

$J_{T_{G_{10}}} = \frac{\pi}{20}\,[0.0000 - 0.4303 + 2(0.0958 + 0.3846 - 0.7760 + 1.0000 - 0.3999 - 0.9924 + 0.0333 +$ $0.9902)]$

$J_{T_{G_{10}}}$=0.7224

2. Comparison

$J_{exacte} = 0.7224$

$\left|J_{exacte} - J_{T_{G_{15}}}\right|$=|$0.7726 - 0.5044$|=0.2682

$\left|J_{exacte} - J_{T_{G_{10}}}\right|$=|$0.7726 - 0.7224$|=0.0502

For the case $n = 5$, the absolute error is 0.2682 compared to the exact solution. For the case $n = 10$, the absolute error has been reduced to 0.0502. This absolute error is about 5 times smaller than the error obtained with 5 intervals.

Exercise03

Calculate the integral using the generalized trapezoidal method.

1. With four small intervals

$$J = \int_0^1 \frac{1}{x+1} dx$$

$$h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$$

x_i	x_0=0	x_1=0.25	x_2=$2\pi/5$	x_3=$3\pi/5$	x_4=$4\pi/5$
$f(x_i)$	1	0.8	0.666667	0.571429	0.5

With:

$\int_a^b f(x)dx \cong J_{T_{G_4}} = \frac{h}{3}\left[f(x_0) + f(x_5) + 4\big(f(x_1) + f(x_3)\big) + f(x_3) + 2f(x_2)\right]$

$J_{S_{G_4}} = \frac{0.25}{3}\,[1 + 0.5 + 4(0.8 + 0.571429) + 2 * 0.666667)] = 0.693254$

2. With 8 small intervals

$$h = \frac{b-a}{n} = \frac{1-0}{8} = \frac{1}{8}$$

x_i	x_0=0	x_1=0.125	x_2=0.250	x_3=0.375	x_4=0.500	x_5=0.625
$f(x_i)$	1	0.888889	0.8	0727273	0.666667	0.615385

x_6=0.750	x_7=0.875	x_8=1.000
0.571429	0.533333	0.5

With:

$\int_a^b f(x)dx \cong J_{SG_8} =$

$J_{SG_8} = \frac{h}{3}\left[f(x_0) + f(x_8) + 4\big(f(x_1) + f(x_2) + f(x_3) + f(x_5) + f(x_7) + 2(f(x_2) + f(x_4) + f(x_6)\big)\right]$

$J_{SG_8} = \frac{0.125}{3}\,[1 + 0.5 + 4(0.888889 + 0727273 + 0.615385 + 0.533333 + 2(0.8 + 0.666667 +$
$0.571429)]$

J_{SG_8}=0.693155

3. Comparison

$J_{exacte} = \int_0^1 \frac{1}{x+1} dx = [\ln(x+1)]_0^1 = ln2$

$|J_{exacte} - J_{SG_4}|$=$|ln2 - 0.693254|$=$1.0682.10^{-4}$.

$|J_{exacte} - J_{SG_8}|$=$|ln2 - 0.693155|$=$7.8194.10^{-6}$.

For the case $n = 4$, the absolute error is $1.0682.10^{-4}$ relative to the exact solution. For the case n = 8 , the absolute error has been reduced to $7.8194.10^{-6}$.This absolute error is approximately 14 times smaller than the error obtained with 4 intervals.

3. Maximal error

$$|\boldsymbol{R_{SG}}| \leq \boldsymbol{E_{max}} = \frac{\boldsymbol{nh^5}}{\boldsymbol{2*90}}\ \boldsymbol{M}$$

With $h = \frac{b-a}{n}$ $,M = Max\{|f^{(4)}(\xi)|\}$, $\boldsymbol{\xi} \in [\boldsymbol{a},\boldsymbol{b}]$

So $\boldsymbol{E_{max}} = \frac{(\boldsymbol{b}-\boldsymbol{a})^5}{180\boldsymbol{n}^4}\ \boldsymbol{M}$

Calculus of M

$f(x) = \frac{1}{x+1}$, $f'(x) = -\frac{1}{(x+1)^2}$, , $f''(x) = 2\frac{1}{(x+1)^3}$, , $f'''(x) = -6\frac{1}{(x+1)^4}$, , $f''''(x) = 24\frac{1}{(x+1)^5}$,

$M = Max\left\{\left|24\frac{1}{(x+1)^5}\right|\right\} = Max\left\{\left|24\frac{1}{(x+1)^5}\right|\right\}$ $, x \in [0,1]$

$\forall x_1 \in [0,1]$, $\forall x_2 \in [0,1]$, $x_1 < x_2 \Rightarrow x_1 + 1 < x_2 + 1 \Rightarrow (x_1 + 1)^5 < (x_2 + 1)^5$

$\Rightarrow \frac{1}{(x_1+1)^5} > \frac{1}{(x_2+1)^5} \Rightarrow \frac{24}{(x_1+1)^5} > \frac{24}{(x_2+1)^5} \Rightarrow |f^{(4)}(x_1)| > |f^{(4)}(x_2)|$

$\Rightarrow |f^{(4)}(x_2)| \searrow$ $\Rightarrow$The maximum is obtained for $x = 0$

$M = 24\frac{1}{(0+1)^5}$=25.

For the case $n = 4,$, we obtain:

$$E_{max} = \frac{(1-0)^5}{180*4^4}\ 24 = 5.2083.10^{-4}$$

Exercise 4:

1. **Calculation of the integral using the generalized trapezoidal method**

$$J = \int_0^{\pi/2} f(x)dx$$

$$h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$$

x_i	x_0=0	x_1=$\pi/8$	x_2=$\pi/4$	x_3=$3\pi/8$	x_4=$\pi/2$
$f(x_i)$	1	0.382683	0.707107	0.923880	1

$$h = x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = x_4 - x_3 = \pi/2$$

$\int_a^b f(x)dx \cong J_{TG_4} = \frac{h}{2}\left[f(x_0) + f(x_4) + 2\big(f(x_1) + f(x_2) + f(x_3)\big)\right]$

$J_{TG_4} = \frac{\pi}{16}$ [0 + 1 + 2(0.382683 + 0.707107 + 0.923880)]=0.987116

2. Calculation of the integral using the generalized Simpson's method

$\int_a^b f(x)dx \cong J_{TG_4} = \frac{h}{3}\left[f(x_0) + f(x_4) + 4\big(f(x_1) + f(x_3)\big) + 2f(x_2)\big)\right]$

$J_{SG_4} = \frac{\pi}{24}$ [0 + 1 + 4(0.382683 + 0.923880) + 2 ∗ 0.707107]=1.000135

3. Comparison

$J_{exacte} = \int_0^{\pi/2} sinx\, dx = -[\cos x]_0^{\pi/2} = 1$

$\left|J_{exacte} - J_{SG_4}\right|$=|1 − 0.987116| =0.012884.

$\left|J_{exacte} - J_{SG_8}\right|$=|1 − 1.000135|=0.000135

It can be observed that the Simpson's method provides better precision than the trapezoidal method.

4. Number of intervals n

The theoretical error committed by the Simpson's method is verified by:

$$|\boldsymbol{R_{SG}}| \le \boldsymbol{E_{max}} = \frac{\boldsymbol{nh^5}}{\boldsymbol{2 * 90}}\ \boldsymbol{M}$$

Where $h = \frac{b-a}{n}$, $M = Max\{\left|f^{(4)}(\xi)\right|\}$, $\boldsymbol{\xi} \in [\boldsymbol{0}, \frac{\boldsymbol{\pi}}{\boldsymbol{2}}]$

Thus, for $|\boldsymbol{R_{SG}}| \le \varepsilon$ it is sufficient that n satisfies

$\frac{nh^5}{180} M = \frac{(b-a)^5}{180n^4} M \le \varepsilon \Rightarrow n^4 \ge \frac{(b-a)^5}{180\varepsilon} M$

$f(x) = sinx \Rightarrow f^{(4)}(x) = sinx$

$f^{(4)}(x) \sim [0, \frac{\pi}{2}] \Rightarrow M = f^{(4)}\left(\frac{\pi}{2}\right) = 1$

Thus $n^4 \ge \frac{\left(\frac{\pi}{2} - 0\right)^5}{180 * 10^{-6}} * 1 \Rightarrow n \ge 15.18$

We take $n = 16$.

Exercise 5 (Supplementary)

1. Calculation of the Integral Using the Generalized Trapezoidal Method

a. With four Small Intervals

$$J = \int_0^1 e^x dx$$

$$h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$$

x_i	x_0=0	x_1=0.25	x_2=0.50	x_3=0.75	x_4=1
$f(x_i)$	1	1.2840	1.6487	2.1170	2.7138

$\int_a^b f(x)dx \cong J_{T_{G_4}} = \frac{h}{2}\left[f(x_0) + f(x_4) + 2\big(f(x_1) + f(x_2) + f(x_3)\big)\right]$

$J_{TG_4} = \frac{0.25}{2}$ [1 + 2.7138 + 2(1.2840 + 1.6487 + 2.7138)]=01.7272

2. With eight small intervals

$$h = \frac{b-a}{n} = \frac{1-0}{8} = \frac{1}{8}$$

x_i	x_0=0	x_1=0.125	x_2=0.250	x_3=0.375	x_4=0.500	x_5=0.625	x_6=0.750	x_7=0.875	x_8=1.000
$f(x_i)$	1	1.1331	1.2840	1.4550	1.6487	1.8682	2.1170	2.3989	2.7183

With:

$$\int_a^b f(x)dx \cong J_{T_{G_8}}$$

$J_{T_{G_8}}=\frac{h}{2}\left[f(x_0) + f(x_8) + 2\big(f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6) + f(x_7)\big)\right]$

$J_{S_{G_8}} = \frac{0.125}{2}$ [1 + 2.7183 + 4(1.1331 + 1.2840 + 1.4550 + 1.6487 + 1.8682 + 2.1170 + 2.3989)]

$J_{S_{G_8}}$=1.7205

3. Comparison

$J_{exacte} = \int_0^1 e^x\, dx = -[e^x]_0^1 = 2.7183$

$\left|J_{exacte} - J_{TG_4}\right|$=$|1.7183 - 0.987116|$ =0.0089.

$\left|J_{exacte} - J_{TG_8}\right|$=$|1.7183 - 1.000135|$=0.0022

For the case $n = 4$, the absolute error is 0.0089 compared to the exact solution. For the case $n = 8$, the absolute error is reduced to 0.0022. This absolute error is approximately 4 times smaller than the error obtained with 4 intervals.

4. Maximum Error

$|\boldsymbol{R_{TG}}| \leq \boldsymbol{E_{max}} = \dfrac{\boldsymbol{nh^3}}{\boldsymbol{12}}\ \boldsymbol{M}$

Where $h = \frac{b-a}{n}$, $M = Max\{|f''(\xi)|\}$, $\boldsymbol{\xi} \in [\boldsymbol{a}, \boldsymbol{b}]$

So $\boldsymbol{E_{max}} = \frac{\boldsymbol{(b-a)^3}}{\boldsymbol{12n^2}}\ \boldsymbol{M}$

Calculus of M

$f(x) = e^x, \quad f'(x) = e^x, \quad f''(x) = e^x,$

$M = Max\{|e^x\,|\} = Max\{e^x\}\ \ , x \in [0,1]$

$(e^x)' = e^x > 0 \Rightarrow f''(x) \nearrow \Rightarrow M = e^1 = e$

For the case $n = 4,$, we obtain:

$$E_{max} = \frac{(1-0)^3}{12 * 4^2}\ e = 0.0142$$

For the case $n = 8,$, we obtain:

$$E_{max} = \frac{(1-0)^3}{12 * 8^2}\ e = 0.0035.$$