Tutorial N.o3

Exercise 1:

1. Rewrite the linear system so that it becomes diagonally dominant:

$$\begin{cases} -2x_1 + 10x_3 = 7 \\ 10x_1 - x_2 = 9 \\ -x_1 + 10x_2 - 2x_3 = 10 \end{cases}$$

2. Using the **Jacobi method**, then the **Gauss-Seidel method**, compute the first 3 iterations by taking

X^0=(0,0,0)^T

Exercise 2:

1.Consider the following linear system:

$$\begin{cases} 3x_1 + x_2 - x_3 = 2 \\ x_1 + 5x_2 + 2x_3 = 17 \\ 2x_1 - x_2 - 6x_3 = -18 \end{cases}$$

Starting from X^0=(0,0,0)^T,compute the first 5 iterations of both the **Jacobi** and **Gauss-Seidel** methods.

2.Knowing that the exact solution is X^0=(1,2,3)^T, what can you conclude?

Exercise 3 (Supplementary): Using the **Jacobi method**, then the **Gauss-Seidel method**, compute the first 3 iterations by taking X^0=(0,0,0)^T

$$\begin{cases} \mathbf{2}x_1 - x_2 + x_3 = -1 \\ 3x_1 - 3x_2 + 9x_3 = 0 \\ 3x_1 + 2x_2 + 5x_3 = 10 \end{cases}$$

2. Why this divergence and which method diverges the fastest.

Solution of Tutorial N.o3

Exercise 1:

1. Rewriting the system to be diagonally dominant:

For the system to be diagonally dominant, the condition must be satisfied:

$$\sum_{\substack{j=1\\ j\neq i}}^{3} |a_{ij}| < |a_{ii}| \ \forall\, i = \overline{1,n}$$

The diagonally dominant system is:

$$\begin{cases} 10x_1 - x_2 = 9 \\ x_1 + 10x_2 - 2x_3 = 10 \\ -2x_1 + 10x_3 = 7 \end{cases}$$

Verification:

$$\begin{cases} |-1| + |0| < |10| \\ |-1| + |-2| < |10| \\ |-1| + |-2| < |10| \end{cases}$$

2. Calculation of the first three iterations:

a. Jacobi Method:

Recursive system:

$$\begin{cases} x_1^{k+1} = \dfrac{x_2^{k} + 9}{10} \\ x_2^{k+1} = \dfrac{x_1^{k} + 2x_3^{k} + 10}{10} \\ x_3^{k+1} = \dfrac{2x_1^{k} + 10}{10} \end{cases}$$

Starting from $X^0 = [0, 0, 0]^T$:

k	0	1	2	3
x_1	0.0000	0.9000	1.0000	1.0230
x_2	0.0000	1.0000	1.2300	1.2760
x_3	0.0000	0.7000	0.8800	0.9000

Approximate solution: $X = [1.0230, 1.2760, 0.9000]^T$

b. Gauss-Seidel Method:

Recursive system:

$$\begin{cases} x_1^{k+1} = \dfrac{x_2^{k} + 9}{10} \\ x_2^{k+1} = \dfrac{x_1^{k+1} + 2x_3^{k} + 10}{10} \\ x_3^{k+1} = \dfrac{2x_1^{k+1} + 10}{10} \end{cases}$$

Starting from $X^0 = [0, 0, 0]^T$:

k	*0*	*1*	*2*	*3*
x_1	*0.0000*	0.9000	*1.0090*	1.0277
x_2	*0.0000*	1.0900	1.2769	1.2831
x_3	0.0000	0.8800	0.9018	0.9055

Approximate solution: $X = [1.0277, 1.2831, 0.9055]^T$

Comparison with Exact Solution:

Exact solution: X = $[507/493 \approx 1.0284,\ 633/493 \approx 1.2840,\ 893/986 \approx 0.9057]^T$

Observation:

For the same number of iterations, the Gauss-Seidel method provides a more accurate solution and converges faster than the Jacobi method.

Exercise 2:

Linear System:

$$\begin{cases} 3x_1 + x_2 - x_3 = 2 \\ x_1 + 5x_2 + 2x_3 = 17 \\ 2x_1 - x_2 - 6x_3 = -18 \end{cases}$$

1. Calculation of the first five iterations:

a. Jacobi Method:

Recursive system:

$$\begin{cases} x_1^{k+1} = \dfrac{-x_2^{k} + -x_3^{k} + 2}{3} \\ x_2^{k+1} = \dfrac{-x_1^{k} - 2x_3^{k} + 17}{5} \\ x_3^{k+1} = \dfrac{2x_1^{k} - x_3^{k} + 18}{6} \end{cases}$$

Starting from $X^0 = [0, 0, 0]^T$:

k	0	1	2	3	4	5
x_1	0.0000	0.6667	0.5333	0.8630	0.8674	0.9406
x_2	0.0000	3.4000	2.0667	2.2311	2.0941	2.0602
x_3	0.0000	3.0000	2.6556	2.8333	2.9158	2.9401

Approximate solution: $X = [0.9406, 2.0602, 2.9401]^T$

b. Gauss-Seidel Method:

Recursive system:

$$\begin{cases} {x_1}^{k+1} = \dfrac{-{x_2}^k + -{x_3}^k + 2}{3} \\ {x_2}^{k+1} = \dfrac{-{x_1}^{k+1} - 2{x_3}^k + 17}{5} \\ {x_3}^{k+1} = \dfrac{2{x_1}^{k+1} - {x_3}^{k+1} + 18}{6} \end{cases}$$

Starting from $X^0 = X = [0,0,0]^T$

k	0	1	2	3	4	5
x_1	0.0000	0.6667	0.4704	0.8498	0.9381	0.99775
x_2	0.0000	3.2667	2.2348	2.1163	2.0402	2.0154
x_3	0.0000	2.6778	2.7843	2.9305	2.9727	2.9899

Approximate solution: Approximate solution: $X = [0.9406, 2.0154, 2.9899]^T$

2. Conclusion:

The exact solution is: $X = [1, 2, 3]^T$

By comparing the approximate solutions to the exact solution, the Gauss-Seidel method provides a more accurate solution after five iterations. The Gauss-Seidel solution $[0.9775, 2.0154, 2.9899]^T$) is closer to the exact solution than the Jacobi solution ($[0.9406, 2.0602, 2.9401]^T$). The Gauss-Seidel method generally converges faster due to its use of the most recent values, though this is not always true for all systems.

Solution to Exercise 3

$$\begin{cases} 3x_1 - x_2 + x_3 = -1 \\ 3x_1 - 3x_2 + 9x_3 = 0 \\ 3x_1 + 2x_2 + 5x_3 = 10 \end{cases}$$

1. Calculation of the first five iterations:

a. Jacobi Method:

Recursive system:

$$\begin{cases} x_1^{k+1} = \dfrac{x_2^k + -x_3^k - 1}{2} \\ x_2^{k+1} = x_1^k + 3x_3^k \\ x_3^{k+1} = \dfrac{10 - 3x_1^k - 2x_3^k}{5} \end{cases}$$

Starting from $X^0 = X = [0,0,0]^T$

k	0	1	2	3
x_1	0.00	-0.50	-1.50	1.10
x_2	0.00	0.00	5.50	5.40
x_3	0.00	2.00	2.30	0.70

b. Gauss-Seidel Method: Recursive system:

$$\begin{cases} x_1^{k+1} = \dfrac{x_2^k + -x_3^k - 1}{2} \\ x_2^{k+1} = x_1^{k+1} + 3x_3^k \\ x_3^{k+1} = \dfrac{10 - 3x_1^{k+1} - 2x_3^{k+1}}{5} \end{cases}$$

Starting from $X^0 = X = [0,0,0]^T$

k	0	1	2	3
x_1	0.00	-0.50	-2.00	1.75
x_2	0.00	-0.50	5.50	4.75
x_3	0.00	2.50	1.00	-.095

Analysis and Conclusion: - The solution diverges because the linear system is not diagonally dominant (the sufficient condition for convergence is not satisfied).

- The table below calculates the relative error for each iteration:

k	$\frac{\left\|\vec{X}^{(k)} - \vec{X}^{(k-1)}\right\|}{\left\|\vec{X}^{(k)}\right\|}$ Jacobi	Gauuse.S
1	100%	100%
2	91.06%	107.07%
3	54.95%	83.30%