Tutorial N.o2

Exercise 1: Solve the following system of equations using the Cramer's method.

$$\begin{cases} 2x_1 + x_2 - 4x_3 = 1 \\ 3x_1 + 3x_2 - 5x_3 = 1 \\ 4x_1 + 5x_2 - 2x_3 = 8 \end{cases}$$

Exercise 2:

Solve the following system of equations using the Gauss method.

$$\begin{cases} 4x_1 + x_2 + 2x_3 = 9 \\ 2x_1 + 4x_2 - x_3 = -5 \\ x_1 + x_2 - 3x_3 = -9 \end{cases}$$

Exercise 3: Consider the following system of equations:

$$\begin{cases} x_1 + 3x_2 + 3x_3 = 0 \\ x_1 + x_2 = 1 \\ 3x_1 + 2x_2 + 6x_3 = 11 \end{cases}$$

1. Solve the system using the Gauss method.
2. Solve the system using the Gauss method with pivoting.

Exercise 4:

Consider the following two matrices:

$$A=\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}, \quad B=\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & -1 & -1 \end{bmatrix}$$

1. Compute the determinant of A and B using the Gauss method.
2. Deduce the determinant of A^{-1}, B^{-1} and $(AB)^{-1}$.

Solution of Tutorial N.o2

Exercise 1: Solving the System of Equations Using the Gaussian Elimination Method

The system can be written in the following matrix form:

$A.\vec{X}=\vec{b}$ avec $A = \begin{bmatrix} 4 & 1 & 2 \\ 2 & 4 & -1 \\ 1 & 1 & -3 \end{bmatrix}$, $\vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 9 \\ -5 \\ -9 \end{bmatrix}$,

The augmented matrix $\tilde{A}$ is defined as:

$$\tilde{A} = \left[\begin{array}{ccc:c} 4 & 1 & 2 & 9 \\ 2 & 4 & -1 & -5 \\ 1 & 1 & -3 & -9 \end{array}\right] \begin{matrix} E_1 \\ E_2, \\ E_3 \end{matrix}$$

Step 1: Eliminate the Elements Below the Diagonal in the First Column Using Row E_1

$E_2 \rightarrow E_2 - \frac{1}{2}E_1$

$E_3 \rightarrow E_3 - \frac{1}{4}E_1$

$$\tilde{A}^{(1)} = \left[\begin{array}{ccc:c} 4 & 1 & 2 & 9 \\ 0 & 7/2 & -1 & -19/2 \\ 0 & 3/4 & -3 & -45/4 \end{array}\right] \begin{matrix} E_1 \\ E_2, \\ E_3 \end{matrix}$$

Step 2: Eliminate the Elements Below the Diagonal in the Second Column Using Row E_2

$E_3 \rightarrow E_3 - \frac{3}{14}E_2$

$$\tilde{A}^{(2)} = \left[\begin{array}{ccc:c} 4 & 1 & 2 & 9 \\ 0 & 7/2 & -2 & -19/2 \\ 0 & 0 & -43/14 & -129/14 \end{array}\right] \begin{matrix} E_1 \\ E_2, \\ E_3 \end{matrix}$$

This results in the following upper triangular matrix:

$-\frac{43}{14}x_3 = -\frac{129}{14} \Rightarrow x_3 = 3$

$\frac{7}{2}x_2 - 2x_3 = \frac{19}{2} \Rightarrow x_3 = -1$

$4x_2 + x_2 + 2x_3 = 9 \Rightarrow x_3 = 1$

Therefore, the solution is:

$$\vec{X} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix},$$

<u>Exercise 2</u>: Solving the system of equations using the Gaussian elimination method

The system can be written in the following matrix form:

$A.\vec{X}=\vec{b}$ avec $A=\begin{bmatrix}1&3&3\\1&1&0\\3&2&6\end{bmatrix}$, $\vec{X}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$, and $\vec{b}=\begin{bmatrix}0\\1\\11\end{bmatrix}$,

$K=1\rightarrow$ Pivot$^{(1)}$= max(1,1,3)=3, so we swap rows 1 and 3 to obtain:

$$\begin{bmatrix}3&2&6\vdots&11\\1&1&0\vdots&1\\1&3&3\vdots&0\end{bmatrix}\rightarrow\begin{bmatrix}3&2&6&\vdots 11\\0&1/3&-2&\vdots -8/3\\0&7/3&1&\vdots -11/3\end{bmatrix}$$

$K=2\rightarrow$ Pivot$^{(2)}$= max$(1/3,\ 7/3)$= $7/3$, so we swap rows 2 and 3 to obtain:

$$\begin{bmatrix}3&2&6&\vdots&11\\0&7/3&0&\vdots&-11/3\\0&1/3&-2&\vdots&-8/3\end{bmatrix}\rightarrow\begin{bmatrix}3&2&6&\vdots 11\\0&7/3&1&\vdots -11/3\\0&0&-15/7&\vdots -15/7\end{bmatrix}$$

This results in the following upper triangular matrix:

$$A.\vec{X}=\vec{b}\Leftrightarrow\begin{cases}x_3=1\\x_2=\frac{3}{7}\left(-\frac{11}{3}-x_3\right)\\x_3=\frac{1}{3}(11-2x_2-6x_3)\end{cases}\Leftrightarrow\begin{cases}x_3=1\\x_2=-2\\x_2=3\end{cases}$$

Exercise 3:

1. Calculation of the determinant of matrices A and B using the Gauss method:

$$\tilde{A}=\begin{bmatrix}2&1&1\\1&2&1\\1&1&3\end{bmatrix}\begin{matrix}E_1\\E_2,\\E_3\end{matrix}$$

Step 1: Eliminate the elements below the diagonal in the first column using Row E_1

$E_2\rightarrow E_2-\frac{1}{2}E_1$

$E_3\rightarrow E_3-\frac{1}{2}E_1$

$$\tilde{A}^{(1)}=\begin{bmatrix}2&1&1\\0&3/2&1/2\\0&1/2&5/2\end{bmatrix}\begin{matrix}E_1\\E_2,\\E_3\end{matrix}$$

Step 2: Eliminate the elements below the diagonal in the second column using row E_2

$E_3\rightarrow E_3-\frac{1}{3}E_2$

$$\tilde{A}^{(2)} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3/2 & 1/2 \\ 0 & 0 & 7/3 \end{bmatrix} \begin{matrix} E_1 \\ E_2, \\ E_3 \end{matrix}$$

The determinant of a triangular matrix is the product of its diagonal elements, so:

$$\det(A) = \det\left(\tilde{A}^{(1)}\right) = \det\left(\tilde{A}^{(2)}\right) = 2 * \frac{3}{2} * \frac{7}{3} = 7$$

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & -1 & -1 \end{bmatrix} \begin{matrix} E_1 \\ E_2 \\ E_3 \end{matrix}$$

Step 1: Elimination of the elements below the diagonal in the first column using row E_1

$E_2 \rightarrow E_2 - 2E_1$
$E_3 \rightarrow E_3 - 3E_1$

$$\tilde{B}^{(1)} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -3 \\ 0 & -7 & -4 \end{bmatrix} \begin{matrix} E_1 \\ E_2 \\ E_3 \end{matrix}$$

Step 2: Elimination of the elements below the diagonal in the second column using row E_2

$$E_3 \rightarrow E_3 - \frac{7}{3} E_2$$

$$\tilde{B}^{(2)} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -3 \\ 0 & 0 & 3 \end{bmatrix} \begin{matrix} E_1 \\ E_2 \\ E_3 \end{matrix}$$

The determinant of a triangular matrix is the product of its diagonal elements, so:

$$\det(B) = \det\left(B^{(1)}\right) = \det\left(\tilde{B}^{(2)}\right) = 1 * (-3) * 3 = -9$$

2. Determinants of $\boldsymbol{A^{-1}}$, $\boldsymbol{B^{-1}}$ $\boldsymbol{and}$ $(\boldsymbol{A}.\boldsymbol{B})^{-1}$

Det(A^{-1})=?

We know that: $A.A^{-1} = I \Rightarrow det(A.A^{-1}) = det(I) \Rightarrow det(A) * det(A^{-1}) = det(I) \Rightarrow det(A^{-1}) = det(I) \,/\, det(A)$

Since $det(I) = 1$, $we\ have\ det(A^{-1}) = 1 \,/\, det(A) = 1 \,/\, 7$

$det(B^{-1})$ = ? ,det(B^{-1}) $= \frac{1}{det(B)} = -1/9$

$det(A.B)^{-1}$ = ? , $(A.B^{-1})$=$B^{-1}A^{-1}$, det$(A.B)^{-1}$=det($B^{-1}A^{-1}$)=det (B^{-1})$.\,det(A^{-1}) = \left(-\frac{1}{9}\right) * \frac{1}{7}$

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