Tutorial N.o1

Exercise 1: Consider the following equation: $x^3 - x - 1 = 0$

1. Show that this equation has a solution in the interval $[1,2]$.
2. Is this solution unique?
3. Compute an approximation of this solution using the bisection method with a precision of 10^{-2}.

Exercise 2: Use the bisection method to approximate the solution of the equation:

$1 - xe^x = 0$ in the interval $[0,1]$ with a precision $\varepsilon = 10^{-3}$.

Exercise 3: Consider the following equation: $x - 0.8 - 0.2 \sin x = 0$

Using the Newton-Raphson method, solve the equation in the interval with a precision

$\varepsilon = 10^{-5}$and initial value and x_0

Exercise 4: We want to evaluate $\sqrt{a}$ using the Newton-Raphson method.

1. Write the recurrence equation.
2. Now, consider $a = 7$ and the interval $[1,4]$.
   a. Verify that the Newton-Raphson method converges to a unique solution.
   b. Compute the first four iterations for both cases: $x_0 = 1$ and $x_0 = 3$.

Exercise 5: Consider the following equation: $\cos x - x = 0$

1. Show that this equation has a solution in the interval $[0,1]$.
2. Find the function $g(x)$ that ensures the convergence of the fixed-point method.
3. Compute an approximate solution with a precision $\varepsilon = 10^{-2}$ and $x_0 = 0.5$.

Exercise 6: We want to solve the equation: $x^3 - x - 1 = 0$ using the fixed-point method in the interval $[1,2]$.

1. Show that the function $x = g(x) = \sqrt[3]{x+1}$ satisfies the conditions for convergence.
2. Compute the approximate solution with a precision $\varepsilon = 10^{-2}$ and $x_0 = 0.5$

Compare the number of iterations obtained with those in Exercise 1. What can be concluded

Solution of Tutorial N.o1

Exercice 1 :

Consider the equation : $f(x) = x^3 - x - 1 = 0$, $x \in [1,2]$

1. ***Prove that the equation has a solution in the interval [1, 2] :***

Let $f(\boldsymbol{x}) = \boldsymbol{x^3 - x - 1}$. Since f is a polynomial, it is continuous on [1 , 2] ,Evaluate at the endpoints:

$f(1) = -1, f(2) = 5$; $f(1).f(2) < 0$, by the Intermediate Value Theorem, there exists $c \in [1, 2]$ such that $f(c) = 0$.

2. ***L'unicité de la solution :***

$f'(x) = 3x^2 - 1 > 0 \ \forall x \in [1,2] \Rightarrow f \nearrow$, so f is strictly increasing.

Since f is monotone, the solution$c$ c is unique.

3. ***La bissection :*** *We apply the bisection method with ε = 10⁻².. We use 3 decimal places, and the number of iterations* $\boldsymbol{n}$*can be calculated in advance using the formula:* $\boldsymbol{n \geq \frac{\frac{b-a}{\varepsilon}}{ln2} = \frac{ln10^2}{ln2} \Rightarrow n = 7\ iterarion}$

a	b		$b - c$	$f(c)$
1.000	2.000	1.500	0.500	+0.875
1.000	1.500	1.250	0.250	−0.297
1.250	1.500	1.375	0.125	+0.225
1.250	1.375	1.312	0.063	−0.054
1.312	1.375	1.343	0.032	+0.079
1.312	1.343	1.327	0.016	+0.010
1.312	1.327	1.319	0.008	−0.024

$b - c = 1.327 - 1.319 = 0.008 \leq \varepsilon = 0.010$ donc on arrête les calculs et la solution est $c \simeq 1.319$ et on peut écrire $c = 1.319 \pm 0.010$.

Exercice 2 :

Using the bisection method, we compute an approximate solution to the equation:

$f(x)$ =1 $- xe^x = 0,\ x \in [0,1]$, $\varepsilon = 10^{-3}$

f is is continuous on [0,1].

$f(0) = 1, f(1) = -1.718;\ \ f(0).f(1) < 0$

Moreover, , $f'(x) = -e^x(1 + x) < 0\ \ \forall\, x\ \in [1,2]$, so f is strictly decreasing.

Since f is monotone, the solution c is unique.

The number of iterations n is:

On prendra 4 chiffres après la virgule et le nombre d'itérations n est :

$$n \geq \frac{ln\frac{b-a}{\varepsilon}}{ln2} = \frac{ln10^3}{ln2} = 9.97\ \Rightarrow n = 10\ \text{ iterations}$$

n	a	b	$C = \frac{a+b}{2}$	$b - c$	$f(x)$
1	0.0000	1.0000	0.5000	0.5000	+0.1756
2	0.5000	1.0000	0.7500	0.2500	−0.5877
3	0.5000	0.7500	0.6250	0.1250	−0.1676
4	0.5000	0.6250	0.5625	0.0625	+0.0128
5	0.5625	0.6250	0.5937	0.0313	−0.0750
6	0.5625	0.5937	0.5781	0.0156	−0.0305
7	0.5625	0.5781	0.5703	0.0078	−0.0087
8	0.5625	0.5703	0.5664	0.0039	+0.0020
9	0.5664	0.5703	0.5683	0.0020	−0.0032
10	0.5664	0.5683	0.5673	0.0010	+0.0004

$b - c = 0.5683 - 0.5673 = 0.0010 \leq \varepsilon = 0.0010$ so we stop the calculations.

The solution is $c \simeq 0.5673$ and we can write $c = 0.5673 \pm 0.0010$.

Exercice 3 :

$f(x) = x - 0.8 - 0.2\sin x = 0$, x= 10^{-5} $x \in [\pi/4, \pi/2]$ –

ε=10 −5 , x_0 = π/4

Convergence study:

The function f is defined on the interval $[\pi/4, \pi/2]$such that:

a) $f(\pi/4) = -0.16$,$f(\pi/2) = 0.57$

b) $f'(x) = 1 - 0.2cos$.
If $f'(x) = 0$, then $1 - 0.2\cos x = 0 \Rightarrow \cos x = 5$, which is impossible since $\cos x \in [-1,1]$.
Thus, $f'(x) > 0\ \forall x \in [\pi/4, \pi/2]$.

c) $f''(x) = 0.2\sin x > 0\ \forall x \in [\pi/4, \pi/2]$

The function f satisfies the three conditions, so the Newton-Raphson method converges to a unique solution.

Recursive formula:

$$\boldsymbol{x_k = x_{k-1} - \frac{f(x_{k-1})}{x_k} = x_{k-1} - \frac{x_{k-1} - 0.8 - 0.2\sin(x_{k-1})}{1 - 0.2\cos(x_{k-1})}}$$

k	x_k	$\lvert x_k - x_{k-1}\rvert$
0	$\frac{\pi}{4}$	—
1	0.967120	0.181722
2	0.964335	0.002785
3	0.964334	0.000001

$|x_k - x_{k-1}|$ ≤|=|0.964334−0.964335|=0.000001≤ ε=10 −5 .

The approximate solution is $c \approx 0.964334$.

Exercise 4:

1. Recursive equation: To find $\sqrt{a}$ solve $x^2 - a = 0$.

Let f(x)= $x^2 - a$.

The Newton-Raphson iteration is:

$$x_k = x_{k-1} - \frac{f(x_{k-1})}{x_k} \Rightarrow x_k = x_{k-1} - \frac{{x_{k-1}}^2 - a}{2x_{k-1}} \Rightarrow x_k = \frac{{x_{k-1}}^2 + a}{2x_{k-1}}$$

2. Case: $a = 7$, interval $[1, 4]$

a.Convergence of the Newton-Raphson method:

For a=7, we have $f(x) = x2 - 7 = 0$.

The function f is defined on [1 ,4] such that:

- $f(1) = -6$,$f(4) = 9$, $f(1) \cdot f(4) < 0$
- $f'(x)=2x>0$ $\forall\, x \in [1, 4]$.
- $f''(x) = 2 > 0$.

The convergence conditions are satisfied, so the Newton-Raphson method converges to a unique solution] $c \in [1,4]$

4. First four iterations: The recursive equation is:

$$x_k = \frac{{x_{k-1}}^2 + a}{2x_{k-1}}$$

For $x_0 = 1$

k	x_k
0	1.0000
1	4.0000
2	2.8750
3	2.6549
4	2.6458

Forr $x_0 = 3$

k	x_k
0	3.0000
1	2.6667
2	2.6458
3	2.6457
4	2.6457

Exercise 5:

$f(x) = cosx - x = 0$

1. Existence of a solution in the interval [0,1]:

f is continuous on [0,1].
$f(0) = 1$, $f(1) = -0.46$; $f(0) \cdot f(1) < 0$.
Moreover, $f'(x) = -sin\, x - 1 < 0\ \forall x \in [0,1]$, so f is strictly decreasing.

Since f f is monotone, the solution c is unique.

2. Function g(x) ensuring convergence of the fixed-point method:

$f(x) = cos\, x - x = 0 \Longrightarrow cos\, x = x$

Let $g(x) = cos\, x$

We verify the conditions for convergence of the fixed-point method:

a) $\boldsymbol{Is\ g([0,1]) \subseteq [0,1]\ g([0,1])\ \backslash subseteq\ [0,1]\ g([0,1]) \subseteq [0,1]}$?

The function $g(x)$ is continuous on [0,1].
For $x \in [0, \pi/2]$, $0 \leq cosx \leq 1$.
Thus, g: [0,1] → [0,1]], and since [0,1] ⊆ $\left[0, \frac{\pi}{2}\right]$,g([0,1])⊆[0,1]

b) $Is\ \mid g'(x) \mid \leq k < 1\ \forall x \in [0,1] \mid$

$g'(x) = -sin\, x \Longrightarrow \mid g'(x) \mid = sin$

Since $(sin\, x)' = cos\, x > 0\ \forall x \in [0,1]$, $sinx$ is increasing, and its maximum is at $x = 1$.
$k = sin(1) \approx 0.84 \Longrightarrow \mid g'(x) \mid \leq k = 0.84 < 1$ .Both conditions are satisfied, so the fixed-point method converges. The recursive formula is:

$$x_k = g(x_{k-1}) = cos(x_{k-1})$$

3. Approximate solution: With 10^{-2}, x_0 =0.5

n	x_k	$\lvert x_k - x_{k-1} \rvert$
0	0.500	—
1	0.878	0.378
2	0.639	0.239
3	0.803	0.164

4	x_k	$\lvert x_k - x_{k-1} \rvert$
5	00.769	0.075
6	0.719	0.050
7	0.752	0.033
8	0.730	0.022
9	0.745	0.015
10	0.735	0.010

The approximate solution is $c \approx 0.735$.

Exercise 6:

$f(x) = x^3 - x - 1 = 0,\ x \in [1,2]$

1. Verification of convergence conditions:

g(x)=$\sqrt[3]{(x+1)}$

We verify two conditions:

a) Is g([1,2])⊆[1,2] ?

The function $g(x)$ is continuous on [1,2].
g′(x)= $\frac{1}{3\sqrt[3]{(x+1)}} > 0\ \forall x \in [1,2], so\ g$ is increasing.
$g(1) = 23 \approx 1.$, $g(2) = 33 \approx 1.44$

Thus, $\forall x \in [1,2],\ 1.26 \le g(x) \le 1.44 \implies g([1,2]) \subseteq [1,2]$

b) Is | $g'(x)$ |$\le k < 1\ \forall x \in [1,2]$?

$$g'(x) = \frac{1}{3\sqrt[3]{(x+1)}} \Rightarrow |g'(x)| = \frac{1}{3\sqrt[3]{(x+1)^2}}$$

$\forall x_1 \in [1,2]$, $\forall x_2 \in [1,2]$: $x_1 < x_2 \Rightarrow \frac{1}{3\sqrt[3]{(x_1+1)^2}} > \frac{1}{3\sqrt[3]{(x_2+1)^2}}$

$$\Rightarrow |g'(x_1)| > |g'(x_2)|$$

Thus, | $g'(x)$ | is decreasing, and its maximum is at $x = 1$

k =| $g'(1)$|= 0.21 < 1

Both conditions are satisfied, so the fixed-point method converges. The recursive formula is:

$$x_k = g(x_{k-1}) = 3\sqrt[3]{x_{k-1} + 1}$$

3. Approximate solution:

n	x_k	$\lvert x_k - x_{k-1}\rvert$
0	1.500	—
1	1.357	0.143
2	1.331	0.019
3	1.326	0.005

The approximate solution is $c \approx 1.326$.

4. Comparison and conclusion:

The number of iterations for the fixed-point method $(n = 3)$ is less than that for the bisection method $(n = 7)$.
We conclude that the fixed-point method converges faster than the bisection method.