Chapter VI:

Interpolation

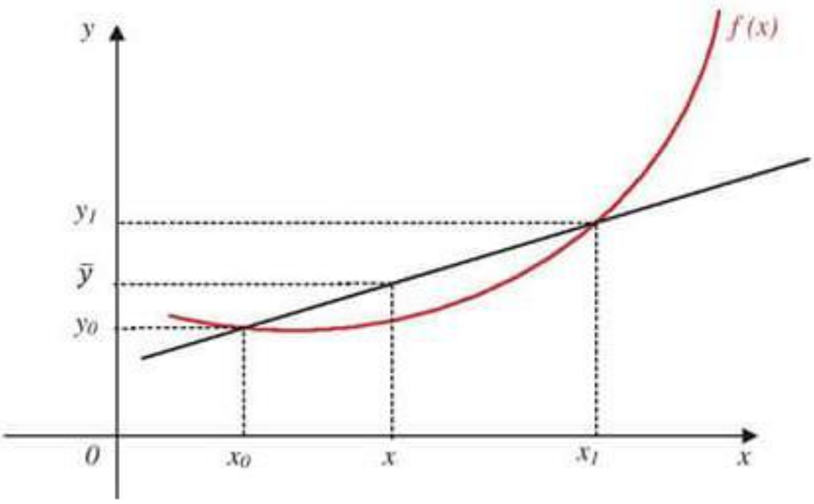
V.1. Introduction

Given a function $f(x)$ known at $(n + 1)$ points of the form $(x_i, f(x_i))$ for $i = 0, 1, 2, \ldots, n$, is it possible to construct an approximation of $f(x)$? Furthermore, for any x, can we derive an approximation using the interpolation points $(x_i(fx_i))$ \)? These points may be obtained from experimental data, a table, or other sources. Even when the function $f(x)$ is not explicitly defined, interpolation enables us to estimate $f(x)$ for values of x distinct from the x_i.

The interpolation problem is relatively straightforward. The solution involves constructing a polynomial of degree at most n that passes through the $n + 1$ interpolation points. This polynomial is referred to as the interpolation polynomial.

VI.2. Lagrange Polynomial

Determine $y(x)$ for $x = 1.5$, given the values $y(1) = 2.7128$ *and* $y(2) = 7.3890$.



Interpolation of a function by a straight line](image_placeholder)

The properties of the straight line within the interval $[x_0, x_1]$ allow us to establish the following relationship:

$$\frac{y_1 - \bar{y}}{x_1 - x} = \frac{y_1 - y_0}{x_1 - x_0}$$

Solving for $\bar{y}$, we obtain:

$$y = \frac{x - x_1}{x_0 - x_1} y_0 + \frac{x - x_0}{x_1 - x_0} y_1$$

This expression can then be used to compute $\bar{y}$ (1.5).

This formula is known as the Lagrange polynomial. In this case, it represents the equation of a straight line, which is a polynomial of degree 1 derived from two measurement points.

VI.2.1 General Formula

It can be demonstrated that the Lagrange polynomial $P_n(x)$ of degree at most n for $n+1$ measurement points is expressed as:

$$\bar{y}=P_n(x)=\frac{(x-x_1)(x-x_2)\ldots\ldots(x_0-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_n)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_1-x_1)(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)}y_1+$$

$$+\cdots\ldots+\frac{(x-x_0)(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_1-x_1)(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)}y_n$$

By defining the Lagrange basis polynomial:

$$L_K(x)=(x-x_0)(x-x_1)\ldots\ldots\ldots(x-x_{K-1})(x-x_{K+1})\ldots\ldots(x-x_n)$$

the simplified form for $(n+1)$ points becomes:

$$P_n(x)=\sum_{K=0}^{n}\frac{L_K(x)}{L_K(x_K)}y_K$$

VI.2.2 Theorem

It can be proven that the Lagrange interpolation polynomial is unique, satisfying the property:

$$\sum_{K=0}^{n}\frac{L_K(x)}{L_K(x_K)}=1$$

Example: Compute the Lagrange polynomial for the following measurement points:

x_i	x_0=0	x_1=1	x_2=2	x_3=3
y_i	y_0=-4	y_1=-2	y_3=3	y_3=14

With 4 points, the degree of the polynomial is at most 3.

$$P_3(x)=\sum_{K=0}^{3}\frac{L_K(x)}{L_K(x_K)}y_K+\frac{L_0(x)}{L_0(x_0)}y_0+\frac{L_1(x)}{L_1(x_1)}y_1+\frac{L_2(x)}{L_2(x_2)}y_2+\frac{L_3(x)}{L_3(x_3)}y_3$$

For k = 0

$$\frac{L_0(x)}{L_0(x_0)} = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_1)(x_0-x_3)} = \frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)} = -\frac{1}{6}(x-1)(x-2)(x-3)$$

For k = 1

$$\frac{L_1(x)}{L_1(x_1)} = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)} = \frac{1}{2}x(x-2)(x-3)$$

For k = 2

$$\frac{L_2(x)}{L_2(x_2)} = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)} = -\frac{1}{2}x(x-1)(x-3)$$

For k = 3

$$\frac{L_3(x)}{L_3(x_3)} = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{(x-0)(x-2)(x-3)}{(2-0)(2-1)(2-3)} = \frac{1}{6}x(x-1)(x-2)(2)$$

Thus:

$$P_n(x) = -\frac{1}{6}(x-1)(x-2)(x-3) + \frac{1}{2}x(x-2)(x-3) - \frac{1}{2}x(x-1)(x-3)(2) + \frac{1}{6}x(x-1)(x-2)$$

2.3. Error Estimation

It can be shown that the error incurred when approximating the original function $f(x)$ with the Lagrange interpolation polynomial P_n(x) over $n + 1$ points is bounded by:

$$|f(x) - P_n(x)| \le \frac{M}{(n+1)!}\prod_{i=0}^{n}|x-x_i|$$

where : $M = Max\left[\left(f^{(n+1)}(\xi)\right)\right],\ \xi \in [x_0,\ x_n]$

VI.3 Newton Method

VI.3 1. General Form

When considering the general representation of a polynomial, the Newton interpolation form is a natural choice, distinct from the standard power basis:

$$P_n(x) = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots \dots \dots . + C_n x^n$$

However, for interpolation purposes, a more suitable form is:

$$P_n(x) = C_0 + C_1(x - x_0) + C_2(x - x_0)(x - x_1) + \cdots \dots . + C_n(x - x_0)(x - x_{n-1})$$

In this form, the coefficient C_n is associated with a product of terms $(x - x_i)$, ensuring that the polynomial is of degree at most n . The key advantage of this form lies in its structure, which facilitates the determination of the $(n + 1)$ coefficients C_i such that the polynomial $P_n(x)$ passes through the $(n + 1)$ interpolation points $(x_i, f(x_i))$. Specifically, we require:

$$P_n(x) = f(x_i) = y_i$$

This condition ensures that the polynomial exactly matches the given data points.

3.2. Relationships Necessary for Calculating the Coefficients C_i

To calculate the coefficients C_i, we use the following relationships:

$$\Delta y_i = y_{i+1} - y_i$$

$$\Delta^2 y_i = \Delta(\Delta y_i) = \Delta(y_{i+1}) - \Delta y_i = y_{i+2} - y_{i+1} + y_i = y_{i+2} - 2y_{i+1} + y_i$$

$$\Delta^3 y_i = \Delta(\Delta^2 y_i) = y_{i+3} - 3y_{i+2} + 3y_{i+1} - y_i$$

More generally, for the k -th differences:

$$\Delta^k y_i = \Delta^{k-1} y_{i+1} - \Delta^k y_{i+1} \qquad (1)$$

$$\Delta^k y_0 = y_k - \binom{k}{1} y_{k-1} + \binom{k}{2} y_{k-2}\Delta + \cdots \dots . + (-1)^k y_0 \qquad (2)$$

$$\binom{i}{1} = C_k^i = \frac{k!}{i!\,(k-i)!}$$

Using the relationship (1), we can construct the following table:

i	x_i	y_i	Δy_i $\Delta y_{i+1} - \Delta y_i$	$\Delta^2 y_i$ $\Delta y_{i+1} - \Delta y_i$	$\Delta^3 y_i$ $\Delta^2 y_{i+1} - \Delta^2 y_i$	$\Delta^4 y_i$ $\Delta^3 y_i - \Delta^3 y_i$
0	x_0	y_0				
			Δy_0			
1	x_1	y_1		$\Delta^2 y_0$		
			Δy_1		$\Delta^3 y_i$	
2	x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_i$
			Δy_2		$\Delta^3 y_i$	
3	x_3	y_3		$\Delta^2 y_2$		
			Δy_3			
4	x_4	y_4				
.	.					
.	.					
.	.					

VI.3 2. Calculation of the Coefficients C_i

We take a polynomial of degree 3 in general. Suppose the points \$x_i$) are equidistant $(x_{i+1} - x_i \ = \ h$, i = $\forall\, i = \overline{0, n-1}$.

$$P_3(x) = C_0 + C_1(x - x_0) + C_2(x - x_0)(x - x_1) + C_3(x - x_0)(x - x_1)(x - x_2)$$

$$\begin{cases} P_3(x_0) = C_0 \\ P_3(x_1) = C_0 + hC_1 \\ P_3(x_2) = C_0 + 2hC_1 + 2h^2C_2 \\ P_3(x_3) = C_0 + hC_1 + 6h^2C_2 + 6h^3C_3 \end{cases}$$

Solving the system gives (using the formula (2)):

$$C_0 = y_0$$
$$C_1 = \Delta^2 y_0 / 2h^2$$
$$C_2 = \Delta^3 y_0 / 6h^3$$

$$P_3(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2h^2}(x - x_0)(x - x_1) + \frac{\Delta^3 y_0}{6h^3}(x - x_0)(x - x_1)(x - x_2)$$

For the general case of a Newton interpolation polynomial of degree n , we obtain the following expression:

$$P_n(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2h^2}(x - x_0)(x - x_1) + .. + \frac{\Delta^n y_0}{n!\,h^n}(x - x_0)(x - x_1)\ldots(x - x_2)$$

Remarks:
- Unlike the Lagrange method, the Newton method requires recalculating the coefficients with each change in the data.
- If the points x_i are not equidistant, h and the successive differences can be replaced by the respective differences between the various x_i values.

Example: Find the Newton polynomial that passes through the following equidistant points:

x_i	x_0=0	x_1=1	x_2=2	x_3=3
y_i	y_0=1	y_1=4	y_3=8	y_3=14

On 4 points where the degree of the polynomial is $n = 3$, the Newton polynomial passing through these 4 points is:

$$P_3(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2h^2}(x - x_0)(x - x_1) + \frac{\Delta^3 y_0}{6h^3}(x - x_0)(x - x_1)(x - x_2)$$

where $h = x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = 1$.

x_0=0 $\rightarrow$ y_0=1

$\Delta y_0 = y_1 - y_1 = 3$

$\Delta^2 y_0 = \Delta y_0 - \Delta y_1 = 1$

x_1=1$\rightarrow$ y_1=4

$\Delta^2 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = 1$

$\Delta y_1 = y_1 - y_1 = 4$

$\Delta^2 y_1 = \Delta y_2 - \Delta y_1 = 2$

x_2=2 $\rightarrow$ y_2=8

$\Delta y_2 = y_1 - y_1 = 6$

x_3=3 $\rightarrow$ y_2=8

The polynomial is:

$$P_n(x) = 1 + \frac{3}{1}(x - 0) + \frac{1}{2}(x - 0)(x - 1) + \frac{1}{6}(x - 0)(x - 1)(x - 2)$$

$$= \frac{1}{6}x^3 + \frac{17}{6}x + 1$$