Chapter IV:
Numerical Integration

IV1. Introduction

In this chapter, we will attempt to cover some numerical methods for calculating integrals: the trapezoidal rule, Simpson's rule, the generalized trapezoidal rule, and the generalized Simpson's rule. In practice, we often resort to these methods because the function to be integrated is generally known only at a finite number of points and is difficult to integrate analytically. These integration techniques aim to approximate the value of the integral using several values of the function to be integrated.

We are therefore interested in estimating the value of an integral of the form:

$$\boldsymbol{J = \int_a^b f(x)dx}$$

$\boldsymbol{f}$ is a continuous function defined on the interval $[\boldsymbol{a}, \boldsymbol{b}]$**.** The approximate value of the integral corresponds to the area $\boldsymbol{A}$ under the curve of $\boldsymbol{f(x)}$.

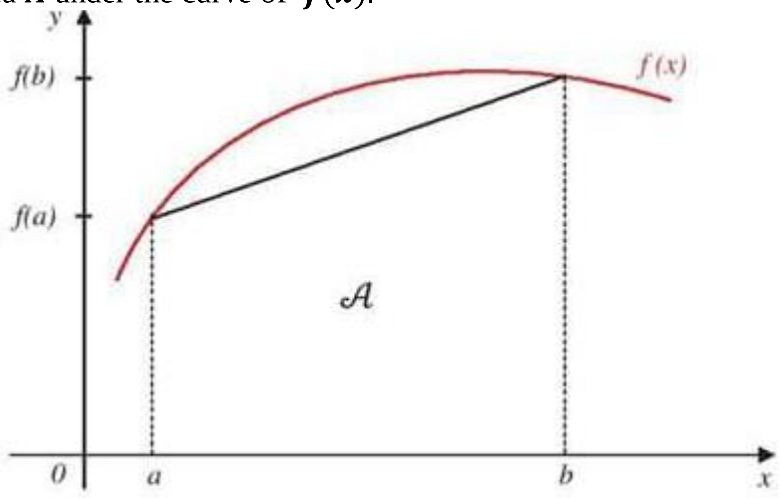


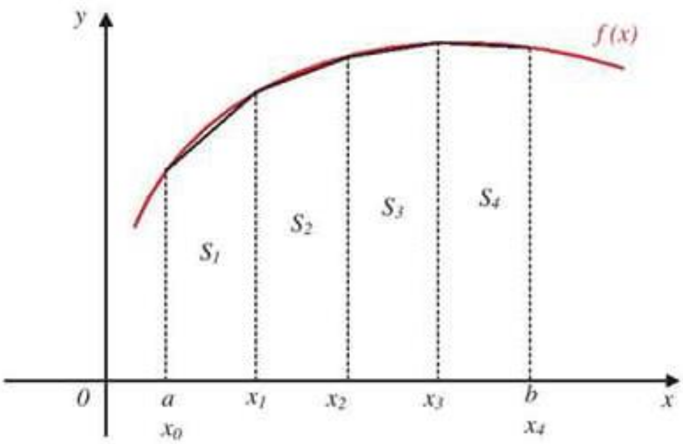
Figure.approximation of the integral by a trapezoid surface

The area corresponds to the surface of the trapezoid given by:

$$S = \frac{\boldsymbol{b-a}}{\boldsymbol{2}}(\boldsymbol{f(a)} + \boldsymbol{f(b)})$$

We can divide the interval $[\boldsymbol{a}, \boldsymbol{b}]$ into n parts and consider the areas of the trapezoids.

Let's take n=4 small intervals:



Approximation of the integral by fourt trapezoid surface.

$\boldsymbol{h = x_{i+1} - x_i \; with \; i = \overline{0, 3}}$

We sum the areas of the 4 trapezoids:

$$\boldsymbol{S = \frac{h}{2}(f(x_i) - f(x_{i+1})}$$

$$\boldsymbol{A = \sum_{i=1}^{4} S_i = \frac{h}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4))}$$

In general, the integral $\boldsymbol{\int_a^b f(x)dx}$can be written as:

$$\boldsymbol{\int_a^b f(x)dx \approx \sum_{i=0}^{n} a_i f(x_i)}$$

where $\boldsymbol{a_i}$are the coefficients to be determined.

4.2. Trapezoidal and Simpson's Method

For $\boldsymbol{n = 1}$: $\boldsymbol{f(x) = 1 \Rightarrow \int_a^b f(x)dx = b - a = a_0 * 1 + a_1 * 1}$

$$\boldsymbol{f(x) = x \Rightarrow \int_a^b xdx = ((b^2)/2) - ((a^2)/2) = a_0 * a + a_1 * b}$$

Solving this system of 2 equations gives: $\boldsymbol{a_0 = a_1 = ((b - a)/2)}$

Thus, we obtain the formula of the trapezoidal rule:

$$\boldsymbol{\int_a^b f(x)dx \approx (b - a)[(1/2)f(a) + f(b)]}$$

The trapezoidal rule can be obtained by approximating f(x) with the straight line segment joining the two points $\boldsymbol{(a, f(a))}$ $\boldsymbol{and}$$\boldsymbol{(b, f(b))}$.

For $\boldsymbol{n=2}$: $\boldsymbol{f(x)=1 \Rightarrow \int_a^b 1dx = b-a = a_0 * 1 + a_1 * 1 + a_2 * 1}$

$$\boldsymbol{f(x)=x \Rightarrow \int_a^b x dx = \left(\frac{b^2}{2}\right) - \left(\frac{a^2}{2}\right) = a^0 * a + a^1 * \left(\left(\frac{a+b}{2}\right)\right) + a^2 * b}$$

$$\boldsymbol{f(x)=x^2 \Rightarrow \int_a^b x^2 dx = ((b^3)/3) - ((a^3)/3) = a_0 * a^2 + a_1 * (((a+b)/2))^2 + a_2 * b^2}$$

Solving this system of 3 equations gives: $\boldsymbol{a_0 = a_2 = ((b-a)/6)}$ et $\boldsymbol{a_1 = 4((b-a)/6)}$

Thus, we obtain Simpson's formula:

$$\boldsymbol{\int_a^b f(x)dx \approx (b-a)\left[\frac{1}{6}f(a) + \frac{4}{6}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f(b)\right]}$$

Simpson's formula can also be obtained by approximating f(x) with the parabola passing through the points $\boldsymbol{(a, f(a)), f(((a+b)/2)))}$ $\boldsymbol{et}$ $\boldsymbol{(b, f(b)}$

3. Generalized Trapezoidal Rule

We divide the interval $\boldsymbol{[a, b]}$ into n equal parts of length $\boldsymbol{h = ((b-a)/n)}$. The resulting subintervals are: $\boldsymbol{[x_0, x_1], [x_1, x_2], \ldots\ldots, [x_{n-1,} x_n]}$

By applying the trapezoidal rule to each subinterval, we get:

$$\boldsymbol{\int_a^b f(x)dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)dx = \sum_{i=0}^{n-1} \frac{h}{2}[f(x_i) + f(x_{i+1})]}$$

We observe that all the terms $\boldsymbol{f(x_i)}$are repeated twice, except the first and the last.

We conclude that:

$$\boldsymbol{\int_a^b f(x)dx = \frac{h}{2}\big[f(x_0) + f(x_n) + 2\big(f(x_1) + f(x_3) + \cdots .. + f(x_n)\big)\big]}$$

This formula is called the generalized trapezoidal rule.

4. Generalized Simpson's Rule

We can improve the accuracy of Simpson's rule by applying it in a composite way.Since Simpson's rule requires an even number of intervals, we divide the integration interval $\boldsymbol{[a, b]}$ into $\boldsymbol{n = 2m}$ subintervals of length $\boldsymbol{h = (b-a)/n}$, and we use the simple Simpson's method in each pair of subintervals $\boldsymbol{[x_{2i}, x_{2i+2} x\_\{2i+2\}], i = 0, 2, \ldots, m-1}$. Then:

$$\boldsymbol{\int_a^b f(x)dx = \sum_{i=0}^{m-1} \int_{x_{2i}}^{x_{2i+2}} f(x)dx = \sum_{i=0}^{m-1} \frac{h}{3}[f(x_{2i}) + 4f(x_{2i+1}) + f(x_{2i+2})]}$$

Note that the odd-indexed terms are multiplied by 4, while the even-indexed terms, except the first and the last, are multiplied by 2. Therefore, we conclude that:

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(x_0) + f(x_n) + 4\big(f(x_1) + f(x_3) + \cdots .. + f(x_{n-1})\big) + 2\big(f(x_2) + f(x_4) + \cdots .. + f(x_{n-2})\big)\right]$$

This formula is called the .generalized Simpson's formula

5. Error Formulas
It is shown that:

Trapezoidal Error

$|R_T| \leq E_{max} = \frac{h^3}{12} M$
where

$$h = b - a \text{ and } M = Max\{|f''(\zeta)|\}, \qquad \zeta \in [a, b]$$

Generalized Trapezoidal Error

$|R_{TG}| \leq E_{max} = \frac{nh^3}{12} M$
where

$$h = \frac{b-a}{n}, \text{and } M = Max\{|f''(\zeta)|\}, \qquad \zeta \in [a, b]$$

Simpson's Error

$|R_S| \leq E_{max} = \frac{h^5}{90} M$
where

$$h = \frac{b-a}{2}, \text{and } M = Max\{|f^{(4)}(\zeta)|\}, \qquad \zeta \in [a, b]$$

Generalized Simpson's Error

$|R_{SG}| \leq E_{max} = \frac{nh^5}{180} M$
where

$$h = \frac{b-a}{2m},, \qquad m = \frac{n}{2}$$

$M = Max\{|f^{(4)}(\zeta)|\}, \ \zeta \in [a, b].$

Example: Consider the following integral:

$$J = \int_0^1 \ln(x+1)\,dx, \quad h = 0.25$$

1. Calculate the integral using Simpson's method.
2. What is the value of h such that the error$\leq 10^{-6}$

$$n = \frac{b-a}{h} = \frac{1-0}{4} = 4$$

$$\int_0^1 \ln(x+1)\,dx = \frac{h}{3}\,[f(0) + f(1) + 2f(0.5) + 4\big(f(0.75) + f(0.25)\big) \approx 0.3862$$

2. $E_{max} = \frac{(b-a)h^4}{180} M$ **,** where $M = Max\{|f^{(4)}(\zeta)|\},\ \zeta \in [a,b]$.

where

$$h = \frac{b-a}{2m},, \qquad m = \frac{n}{2}$$

$f(x)$= ln(x+1)⇒ $\left|f^{(4)}(x)\right| = \left|\frac{-6}{(x+1)^4}\right| = \frac{6}{(x+1)^4}$, $max\left|f^{(4)}(x)\right| = 6$

E$\leq 10^{-6} \Rightarrow \frac{h^4}{180}.6 \leq 10^{-6} \Rightarrow h^4 \leq 30.10^{-3}$

$$h \leq 0.4161$$