Chapter III :
Solving Systems of Linear Equations (Iterative Methods)

III.1. Introduction

We are interested in solving the linear system:

$$A\vec{X} = \vec{b}$$

A is a matrix of order n. The methods we present below are generalizations to the n- dimensional case of the resolution methods of $f(x) = 0$ studied in the one-dimensional case in Chapter 1. These methods involve using an initial vector $\vec{X}^{(0)}$ and generating a sequence of vectors: $\vec{X}^{(0)}$, $\vec{X}^{(1)}$......., $\vec{X}^{(n)}$ that converge as quickly as possible to the solution vector.

3.2 Principle We want to solve the linear system:

$$A\vec{X} = \vec{b}$$

Writing A in the form $A = M - N$ where A and M must be regular, the system can be

rewritten as: (M-N) $\vec{X}$=$\vec{b}$ or M $\vec{X}$=N $\vec{X}$+$\vec{b}$

Thus, $\vec{X}$=M^{-1}N$\vec{X}$+ $M^{-1}\vec{b}$

Starting from the initial vector $\vec{X}^{(0)}$we obtain the iterations $\vec{X}^{(k)}$as follows:

$$\vec{X}=M^{-1}N\vec{X}+ M^{-1}\vec{b} \quad ,k=0,1,2,\ldots\ldots$$

3.3 Decomposition of Matrix

We define the matrices $D, L,$ and U as follows:

- D Diagonal matrix where / d_{ii}=a_{ii}, $\forall\ i$.

- L : Lower triangular matrix where : $\begin{cases} l_{ii} = -a_{ii} & if\ \ i > j \\ l_{ii} = 0 & if\ \ i \leq j \end{cases}$

-U : Upper triangular matrix where:L : $\begin{cases} u_{ij} = -a_{ij} & if\ \ j > i \\ u_{ii} = 0 & if\ \ j \leq i \end{cases}$

From this, we obtain the relation:

$$A = D - L - U$$

3.4 Jacobi Method

Since matrix A is decomposed as:

$$A = M - N$$

Taking $M = D$ and $N = L + U$, relation (1) becomes:

$$\vec{X}^{(k+1)} = D^{-1}(L + U)\vec{X}^{(k)} + D^{-1}\vec{b} \quad (2)$$

which can be expanded as follows:

$$\begin{cases} \vec{X}_1^{(k+1)} = (b_1 - a_{12}\vec{X}_2^{(k)} - a_{13}\vec{X}_3^{(k)} - \cdots - a_{1n}\vec{X}_n^{(k)})/a_{11} \\ \vec{X}_2^{(k+1)} = (b_2 - a_{21}\vec{X}_2^{(k)} - a_{23}\vec{X}_3^{(k)} - \cdots - a_{2n}\vec{X}_n^{(k)})/a_{22} \\ \quad\quad\quad\quad\quad\quad . \\ \vec{X}_n^{(k+1)} = (b_n - a_{n1}\vec{X}_2^{(k)} - a_{n2}\vec{X}_3^{(k)} - \cdots - a_{nn}\vec{X}_{n-1}^{(k)})/a_{nn} \end{cases}$$

3.5 .Gauss-Seidel Method

Since matrix A is decomposed as: $A = M - N$

Taking $M = D - L$ and $N = U$, relation (1) becomes:

$$\vec{X}^{(k+1)} = (D - L)^{-1}U\vec{X}^{(k)} + (D - L)^{-1}\vec{b}$$

Since computing the inverse of $(D - L)$ can be complex, we prefer to write the system as follows:

$$\vec{X}^{(k+1)} = D^{-1}L\vec{X}^{(k+1)} + D^{-1}U\,\vec{X}^{(k)} + D^{-1}\vec{b} \quad (3)$$

By expanding this vector recurrence, we obtain:

$$\begin{cases} \vec{X}_1^{(k+1)} = (b_1 - a_{12}\vec{X}_2^{(k)} - a_{13}\vec{X}_3^{(k)} - \cdots - a_{1n}\vec{X}_n^{(k)})/a_{11} \\ \vec{X}_2^{(k+1)} = (b_2 - a_{21}\vec{X}_2^{(k+1)} - a_{23}\vec{X}_3^{(k)} - \cdots - a_{2n}\vec{X}_n^{(k)})/a_{22} \\ . \\ \vec{X}_n^{(k+1)} = (b_n - a_{n1}\vec{X}_2^{(k+1)} - a_{n2}\vec{X}_3^{(k+1)} - \cdots - a_{nn}\vec{X}_{n-1}^{(k+1)})/a_{nn} \end{cases}$$

The Gauss-Seidel method is thus an improved variant of the Jacobi method. In fact, at iteration $k+1$, when computing$\vec{X}_2^{(k+1)}$, we already have a better approximation of $\vec{X}_1^{(k)}$than $\vec{X}_1^{(k+1)}$, as we have just computed . Similarly, when computing$\vec{X}_3^{(k+1)}$, we can use the newly computed $\vec{X}_1^{(k+1)}$and$\vec{X}_2^{(k+1)}$. More generally, for computing$\vec{X}_i^{(k+1)}$, we can use $\vec{X}_1^{(k+1)}$},$\vec{X}_2^{(k+1)}$,..., $\vec{X}_i^{(k+1)}$ from the previous iteration.

stopping test:

$$|\vec{X}_i^{(n)} - \vec{X}_i^{(n-1)}| \le \varepsilon \ , \forall\, i = \overline{1,n}$$

convergence condition

It can be shown that if A is a strictly diagonally dominant matrix (a sufficient condition), the Jacobi and Gauss-Seidel methods are convergent

$$\sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \le |a_{ii}|, \quad \forall\, i = \overline{1,n}$$

Example 1 : Solve the system

$$\begin{cases} 5x_1 + 2x_2 - x_3 = 6 \\ x_1 + 6x_2 - 3x_3 = 4 \\ 2x_1 + x_2 + 4x_3 = 7 \end{cases}$$

of equations using the Jacobi and Gausse Seidal methods from $X^{(0)} = (0,0,0)^t$^do only the first 3 iterations).

we have A= $\begin{pmatrix} 5 & 2 & -1 \\ 1 & 6 & -3 \\ 2 & 1 & 4 \end{pmatrix}$,

It can be shown that if A is a strictly diagonally dominant matrix : $\sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \leq |a_{ii}|$, ∀i=1,n

$(|2| + |1| \leq |5|,\ \ |1| + |-3| \leq |6|,\ |2| + |1| \leq |4|)$, The condition is satisfied, and therefore the matrix A is diagonally dominant.

1.method Jacobi

$$\begin{cases} 5x_1 + 2x_2 - x_3 = 6 \\ x_1 + 6x_2 - 3x_3 = 4 \\ 2x_1 + x_2 + 4x_3 = 7 \end{cases} \Rightarrow \begin{cases} \vec{x}_1^{\,(k+1)} = \frac{1}{5}(6 - 2\vec{x}_2^{\,(k)} + \vec{x}_3^{\,(k)}) \\ \vec{x}_2^{\,(k+1)} = \frac{1}{6}(4 - \vec{x}_1^{\,(k)} + 3\vec{x}_3^{\,(k)}) \\ \vec{x}_3^{\,(k+1)} = \frac{1}{4}(7 - 2\vec{x}_1^{\,(k)} - \vec{x}_1^{\,(k)}) \end{cases}$$ Tapez une équation ici.

When $k = 0$, $X^{(0)}$=(0,0,0) ^{t}

$$\Leftrightarrow \begin{cases} \vec{x}_1^{\,(1)} = \frac{1}{5}\left(6 - 2\vec{x}_2^{\,(0)} + \vec{x}_3^{\,(0)}\right) = \frac{6}{5} = 1.200 \\ \vec{x}_2^{\,(1)} = \frac{1}{6}\left(4 - \vec{x}_1^{\,(0)} + 3\vec{x}_3^{\,(0)}\right) = \frac{4}{6} = 0.667 \\ \vec{x}_3^{\,(1)} = \frac{1}{4}\left(7 - 2\vec{x}_1^{\,(0)} - \vec{x}_1^{\,(0)}\right) = \frac{7}{4} = 1.750 \end{cases}$$

When $k = 1$,

$$\Leftrightarrow\begin{cases}\vec{x}_1{}^{(2)} = \frac{1}{5}\left(6 - 2\vec{x}_2{}^{(1)} + \vec{x}_3{}^{(1)}\right) = \frac{1}{5}(6 - 2(0.667) + 1.750) = 1.283\\ \vec{x}_2{}^{(2)} = \frac{1}{6}\left(4 - \vec{x}_1{}^{(1)} + 3\vec{x}_3{}^{(1)}\right) = \frac{1}{6}(4 - 1.200 + 3(1.750)) = 1.342\\ \vec{x}_3{}^{(2)} = \frac{1}{4}\left(7 - 2\vec{x}_1{}^{(1)} - \vec{x}_1{}^{(1)}\right) = \frac{1}{4}(7 - 2(1.200) - 0.667) = 0.983\end{cases}$$

When $k = 2$,

$$\Leftrightarrow\begin{cases}\vec{x}_1{}^{(3)} = \frac{1}{5}\left(6 - 2\vec{x}_2{}^{(2)} + \vec{x}_3{}^{(2)}\right) = \frac{1}{5}(6 - 2(1.342) + 0.983) = 0.860\\ \vec{x}_2{}^{(3)} = \frac{1}{6}\left(4 - \vec{x}_1{}^{2} + 3\vec{x}_3{}^{(2)}\right) = \frac{1}{6}(4 - 1.283 + 3(0.983)) = 0.994\\ \vec{x}_3{}^{(3)} = \frac{1}{4}\left(7 - 2\vec{x}_1{}^{(2)} - \vec{x}_1{}^{(2)}\right) = \frac{1}{4}(7 - 2(1.283) - 1.342) = 0.773\end{cases}$$

$$\mathbf{X}=\begin{pmatrix}0.860\\0.994\\0.773\end{pmatrix}$$

2. Gausse Seidal method

$$\begin{cases}5x_1 + 2x_2 - x_3 = 6\\ x_1 + 6x_2 - 3x_3 = 4\\ 2x_1 + x_2 + 4x_3 = 7\end{cases} \Rightarrow \begin{cases}\vec{x}_1{}^{(k+1)} = \frac{1}{5}(6 - 2\vec{x}_2{}^{(k)} + \vec{x}_3{}^{(k)})\\ \vec{x}_2{}^{(k+1)} = \frac{1}{6}(4 - \vec{x}_1{}^{(k+1)} + 3\vec{x}_3{}^{(k)})\\ \vec{x}_3{}^{(k+1)} = \frac{1}{4}(7 - 2\vec{x}_1{}^{(k+1)} - \vec{x}_2{}^{(k+1)})\end{cases}$$ Tapez une équation ici.

When $k = 0$, $X^{(0)}$=(0,0,0) ^{t}

$$\Leftrightarrow\begin{cases}\vec{x}_1{}^{(1)} = \frac{1}{5}\left(6 - 2\vec{x}_2{}^{(0)} + \vec{x}_3{}^{(0)}\right) = \frac{6}{5} = 1.200\\ \vec{x}_2{}^{(1)} = \frac{1}{6}\left(4 - \vec{x}_1{}^{(1)} + 3\vec{x}_3{}^{(0)}\right) = \frac{1}{6}(4 - 1.200 + 3(0)) = 0.467 \quad ,\\ \vec{x}_3{}^{(1)} = \frac{1}{4}\left(7 - 2\vec{x}_1{}^{(1)} - \vec{x}_1{}^{(1)}\right) = \frac{1}{4}(7 - 2(1.200) - 0.467) = 1.033\end{cases}$$

$$\boldsymbol{X}^{(1)}=\begin{pmatrix}0.860\\0.994\\0.773\end{pmatrix}$$

When $k = 1$,

$$\Leftrightarrow\begin{cases}\vec{x}_1{}^{(2)} = \frac{1}{5}\left(6 - 2\vec{x}_2{}^{(1)} + \vec{x}_3{}^{(1)}\right) = \frac{1}{5}(6 - 2(0.467) + 1.033) = 1.220\\ \vec{x}_2{}^{(2)} = \frac{1}{6}\left(4 - \vec{x}_1{}^{(2)} + 3\vec{x}_3{}^{(1)}\right) = \frac{1}{6}(4 - 1.220 + 3(1.033)) = 0.980,\\ \vec{x}_3{}^{(1)} = \frac{1}{4}\left(7 - 2\vec{x}_1{}^{(2)} - \vec{x}_1{}^{(2)}\right) = \frac{1}{4}(7 - 2(1.220) - 0.980) = 0.895\end{cases}$$

$$\boldsymbol{X}^{(2)}=\begin{pmatrix}1.220\\0.980\\0.895\end{pmatrix}$$

When $k = 2$,

$$\Leftrightarrow\begin{cases}\vec{x}_1{}^{(3)} = \frac{1}{5}\left(6 - 2\vec{x}_2{}^{(2)} + \vec{x}_3{}^{(2)}\right) = \frac{1}{5}(6 - 2(0.980) + 0.895) = 0.987\\ \vec{x}_2{}^{(3)} = \frac{1}{6}\left(4 - \vec{x}_1{}^{(3)} + 3\vec{x}_3{}^{(2)}\right) = \frac{1}{6}(4 - 0.987 + 3(0.895)) = 0.950,\\ \vec{x}_3{}^{(3)} = \frac{1}{4}\left(7 - 2\vec{x}_1{}^{(3)} - \vec{x}_1{}^{(3)}\right) = \frac{1}{4}(7 - 2(0.987) - 0.950) = 1.019\end{cases}$$

$$\boldsymbol{X}^{(3)}=\begin{pmatrix}0.987\\0.950\\1.019\end{pmatrix}$$

Example 2 : Solve the system

$$\begin{cases}6\mathrm{x}_1 + \mathrm{x}_2 + \mathrm{x}_3 = 12\\ 2\mathrm{x}_1 + 4\mathrm{x}_2 = 0\\ \mathrm{x}_1 + 2\mathrm{x}_2 + 6\mathrm{x}_3 = 6\end{cases}$$

of equations using the Jacobi and Gausse Seidal methods from $X^{(0)} = (2,2,2)^t$^do only the first 3 iterations).

we have A= $\begin{pmatrix}6 & 1 & 1\\2 & 4 & 0\\1 & 2 & 6\end{pmatrix}$, It can be shown that if A is a strictly diagonally dominant matrix : $\sum_{\substack{j=1\\j\neq i}}^{n}\left|a_{ij}\right| \leq |a_{ii}|$, ∀i=1,n

$(|2|+|1| \leq |5|,\ \ |1|+|-3| \leq |6|,\ |2|+|1| \leq |4|)$, The condition is satisfied, and therefore the matrix A is diagonally dominant.

1.method Jacobi

$$A = D - E - F$$

A= $\begin{pmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}$, E= $\begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ -1 & -2 & 0 \end{pmatrix}$, F= $\begin{pmatrix} 0 & -1 & -1 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

$$\begin{cases} 6x^1 + 2x^2 + x_3 = 12 \\ 2x_1 + 4x_2 = 0 \\ x_1 + x_2 + 6x_3 = 6 \end{cases} \Rightarrow \begin{cases} \vec{x}_1^{\ (k+1)} = \dfrac{1}{6}(12 - \vec{x}_2^{\ (k)} - \vec{x}_3^{\ (k)}) \\ \vec{x}_2^{\ (k+1)} = \dfrac{1}{4}(2\vec{x}_1^{\ (k)}) \\ \vec{x}_3^{\ (k+1)} = \dfrac{1}{6}(6 - \vec{x}_1^{\ (k+1)} - 2\vec{x}_2^{\ (k+1)}) \end{cases}$$

When $k = 0$, $X^{(0)}$=$(2,2,2)^{\ t}$

$$\Leftrightarrow \begin{cases} \vec{x}_1^{\ (1)} = \frac{1}{6}(12 - \vec{x}_2^{\ (0)} - \vec{x}_3^{\ (0)}) = \frac{4}{3} \\ \vec{x}_2^{\ (1)} = \frac{1}{2}\left(\vec{x}_1^{\ (0)}\right) = -1 \\ \vec{x}_3^{\ (1)} = \frac{1}{6}\left(6 - \vec{x}_1^{\ (0)} - 2\vec{x}_1^{\ (0)}\right) = 0 \end{cases}$$

$\boldsymbol{X}^{(\mathbf{1})}$= $\begin{pmatrix} \frac{4}{3} \\ -1 \\ 0 \end{pmatrix}$, $\boldsymbol{X}^{(\mathbf{2})}$= $\begin{pmatrix} \frac{13}{6} \\ -\frac{2}{3} \\ \frac{10}{9} \end{pmatrix}$, $\boldsymbol{X}^{(\mathbf{3})}$= $\begin{pmatrix} \frac{52}{27} \\ -\frac{13}{12} \\ \frac{31}{36} \end{pmatrix}$

Example 3 : Solve the system

$$\begin{cases} 2x_1 - \mathrm{x}_2 = 1 \\ -x_2 + 2\mathrm{x}_3 = 1 \\ -x_1 + 2x_2 - \mathrm{x}_3 = 0 \end{cases}, \varepsilon = 10^{-1} = 0.1 \ , X^{(0)} = (0,0,0)^t$$

we have A= $\begin{pmatrix} 2 & -1 & 0 \\ 0 & -1 & 2 \\ -1 & 2 & -1 \end{pmatrix}$, It can be shown that if A is a strictly diagonally dominant matrix $\sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \leq |a_{ii}|$, ∀i=1,n.

(|-1|+|0|≤|2|, |2|+|0|≤|-1|, |2|+|-1|≤|-1|), The condition is not satisfied, and therefore the matrix A is diagonally dominant.

1.method Jacobi

$$\begin{cases} 2x_1 - \mathrm{x}_2 = 1 \\ -x_1 + 2x_2 - \mathrm{x}_3 = 0 \\ -x_2 + 2\mathrm{x}_3 = 1 \end{cases} \Rightarrow \begin{cases} \vec{x}_1^{\,(k+1)} = \frac{1}{2}(1 + \vec{x}_2^{\,(k)}) \\ \vec{x}_2^{\,(k+1)} = \frac{1}{2}(\vec{x}_1^{\,(k)} + \vec{x}_3^{\,(k)}) \\ \vec{x}_3^{\,(k+1)} = \frac{1}{2}(1 + 2\vec{x}_2^{\,(k)}) \end{cases}$$ Tapez une équation ici.

When $k = 0$, $X^{(0)}$=(0,0,0) ^{t}

$$\Leftrightarrow \begin{cases} \vec{x}_1^{\,(1)} = \frac{1}{2}(1+0) = \frac{1}{2} = 0.500 \\ \vec{x}_2^{\,(1)} = \frac{1}{2}(0+0) = 0 \\ \vec{x}_3^{\,(1)} = \frac{1}{2}(1+0) = \frac{1}{2} = 0.500 \end{cases}$$

$\boldsymbol{X}^{(1)}=\begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}$, $\boldsymbol{X}^{(2)}=\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$, $\boldsymbol{X}^{(3)}=\begin{pmatrix} \frac{3}{4} \\ \frac{1}{2} \\ \frac{3}{4} \end{pmatrix}$, $\boldsymbol{X}^{(4)}=\begin{pmatrix} \frac{3}{4} \\ \frac{3}{4} \\ \frac{3}{4} \end{pmatrix}$, $\boldsymbol{X}^{(5)}=\begin{pmatrix} \frac{7}{8} \\ \frac{3}{4} \\ \frac{7}{8} \end{pmatrix}$, $\boldsymbol{X}^{(6)}=\begin{pmatrix} \frac{7}{8} \\ \frac{7}{8} \\ \frac{7}{8} \end{pmatrix}$,

$\boldsymbol{X}^{(7)}=\begin{pmatrix} \frac{15}{16} \\ \frac{7}{8} \\ \frac{15}{16} \end{pmatrix}$, $\boldsymbol{X}^{(8)}=\begin{pmatrix} \frac{15}{16} \\ \frac{15}{16} \\ \frac{15}{16} \end{pmatrix}$.

$|\vec{x}_i^{(k+1)}$ -$\vec{x}_i^{(k)}| \leq \varepsilon$, i= $\overline{1, n}$

$|\vec{x}_i^{(8)}$ -$\vec{x}_i^{(7)}| \leq \varepsilon$, i= $\overline{1, n}$: $\begin{cases} |\vec{x}_1^{(8)} - \vec{x}_1^{(7)} = |\frac{15}{16} - \frac{15}{16}| = 0 < \varepsilon \\ |\vec{x}_2^{(8)} - \vec{x}_2^{(7)} = |\frac{15}{16} - \frac{7}{8}| = 0{,}05 < \varepsilon\, 15 \\ |\vec{x}_3^{(8)} - \vec{x}_3^{(7)} = |\frac{15}{16} - \frac{15}{16}| = 0 < \varepsilon \end{cases}$

Example 4 Solve the system with Gausse Seidal method

$$\begin{cases} 4x_1 + \mathrm{x}_2 - x_3 = 3 \\ 2x_2 + 7x_2 + \mathrm{x}_3 = 19 \\ x_1 - 3x_2 + 12\mathrm{x}_3 = 31 \end{cases}, \varepsilon = 10^{-1} = 0.1 \ , X^{(0)} = (0,0,0)^t$$

we have A= $\begin{pmatrix} 4 & -1 & -1 \\ 2 & 7 & 1 \\ 1 & -3 & 12 \end{pmatrix}$, It can be shown that if A is a strictly diagonally dominant matrix $\sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \leq |a_{ii}|$, ∀i=1,n.

(|1|+|-1|≤|4|, |2|+|1|≤|7|, |1|+|-3|≤|12|), The condition is satisfied, and therefore the matrix A is diagonally dominant.

Gausse Seidal method

$$\begin{cases} 4x_1 + \mathrm{x}_2 - x_3 = 3 \\ 2x_2 + 7x_2 + \mathrm{x}_3 = 19 \\ x_1 - 3x_2 + 12\mathrm{x}_3 = 31 \end{cases} \Rightarrow \begin{cases} \vec{x}_1^{(k+1)} = \frac{1}{4}\left(3 - \vec{x}_2^{(k)} + \vec{x}_3^{(k)}\right) \\ \vec{x}_2^{(k+1)} = \frac{1}{7}\left(19 - \overrightarrow{2x_1}^{(k+1)} - 3\vec{x}_3^{(k)}\right) \\ \vec{x}_3^{(k+1)} = \frac{1}{12}\left(31 - \vec{x}_1^{(k+1)} + 3\,\vec{x}_1^{(k+1)}\right) \end{cases}$$

When $k = 1$, $X^{(0)}$=(0,0,0) ^{t}

$$\Leftrightarrow \begin{cases} \vec{x}_1^{(1)} = \frac{1}{4}\left(3 - \vec{x}_2^{(0)} + \vec{x}_3^{(0)}\right) = \frac{1}{4}(3 - 0 + 0) = \frac{3}{4} = 0.75 \\ \vec{x}_2^{(1)} = \frac{1}{7}\left(19 - \overrightarrow{2x_1}^{(1)} - 3\vec{x}_3^{(0)}\right) = \frac{1}{7}(19 - 2(0.75) - 0) = 2.5, \\ \vec{x}_3^{(1)} = \frac{1}{12}\left(31 - \vec{x}_1^{(1)} + 3\,\vec{x}_1^{(1)}\right) = \frac{1}{12}(31 - 0.75 + 3(2.5)) = 3.15 \end{cases} , \boldsymbol{X}^{(\mathbf{1})} = \begin{pmatrix} \frac{3}{4} \\ \mathbf{2.5} \\ \mathbf{3.1} \end{pmatrix}$$

$$X^{(1)}=\begin{pmatrix}0.75\\ \mathbf{2.5}\\ \mathbf{3.15}\end{pmatrix}$$

When $k = 1$,

$$\Leftrightarrow\begin{cases} \vec{x}_1{}^{(2)} = \frac{1}{4}\left(3 - \vec{x}_2{}^{(1)} + \vec{x}_3{}^{(1)}\right) = \frac{1}{4}(3 - 2.5 + 3.15) = 0.91 \\ \vec{x}_2{}^{(2)} = \frac{1}{7}\left(19 - \overrightarrow{2x_1}{}^{(2)} - 3\vec{x}_3{}^{(1)}\right) = \frac{1}{7}(19 - 2(0.91) - 3.15) = 2.00 \\ \vec{x}_3{}^{(2)} = \frac{1}{12}\left(31 - \vec{x}_1{}^{(2)} + 3\,\vec{x}_1{}^{(2)}\right) = \frac{1}{12}(31 - 0.91 + 3(2.00)) = 3.01 \end{cases}$$

$$X^{(2)}=\begin{pmatrix}0.91\\ \mathbf{2.00}\\ \mathbf{3.01}\end{pmatrix}$$

When $k = 2$,

$$\Leftrightarrow\begin{cases} \vec{x}_1{}^{(3)} = \frac{1}{4}\left(3 - \vec{x}_2{}^{(2)} + \vec{x}_3{}^{(2)}\right) = \frac{1}{4}(3 - 2.00 + (3.01)) = 1{,}00 \\ \vec{x}_2{}^{(3)} = \frac{1}{7}\left(19 - \overrightarrow{2x_1}{}^{(3)} - 3\vec{x}_3{}^{(2)}\right) = \frac{1}{7}(19 - 2(1.00) - 3.01) = 2.00 \\ \vec{x}_3{}^{(3)} = \frac{1}{12}\left(31 - \vec{x}_1{}^{(3)} + 3\,\vec{x}_1{}^{(3)}\right) = \frac{1}{12}(31 - 1.00 + 3(2.00)) = 3.00 \end{cases}$$

$$X^{(3)}=\begin{pmatrix}1.00\\ 2.00\\ 3.00\end{pmatrix}$$

When $k = 3$,

$$\begin{cases} \vec{x}_1{}^{(4)} = \frac{1}{4}\left(3 - \vec{x}_2{}^{(3)} + \vec{x}_3{}^{(3)}\right) = \frac{1}{4}(3 - 2.00 + (3.01)) = 1{,}00 \\ \vec{x}_2{}^{(4)} = \frac{1}{7}\left(19 - \overrightarrow{2x_1}{}^{(4)} - 3\vec{x}_3{}^{(3)}\right) = \frac{1}{7}(19 - 2(1.00) - 3.00) = 2.00 \\ \vec{x}_3{}^{(4)} = \frac{1}{12}\left(31 - \vec{x}_1{}^{(4)} + 3\,\vec{x}_1{}^{(4)}\right) = \frac{1}{12}(31 - 1.00 + 3(2.00)) = 3.00 \end{cases}$$

$$X^{(4)}=\begin{pmatrix}1.00\\ 2.00\\ 3.00\end{pmatrix}$$

so the approximate solution $X^{(4)}=\begin{pmatrix}1.00\\ 2.00\\ 3.00\end{pmatrix}$