Chapter II:
Solving Systems of Linear Equations (Direct Methods)

Introduction

In practice, engineers often face problems whose solution involves solving a system of equations that models the various elements considered. For example, determining currents and voltages in electrical networks requires solving a system of linear or nonlinear equations.

In this chapter, we will discuss two main methods for solving linear systems, namely Gaussian elimination and Gaussian elimination with pivoting. Generally, solving a system of linear equations consists of finding a vector: $X = [x_1 , x_2, \ldots\ldots, x_n]^T\}$ ((where T denotes the transpose, making it a column vector), that satisfies:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots + a_{2n}x_n = b_2 \\ . \\ a_{1n}x_1 + a_{2n}x_2 + a_{3n}x_3 + + \ldots\ldots + a_{nn}x_n = b_n \end{cases}$$

We can use matrix notation, which is much more practical and compact. The system above can be rewritten as:

$$A\vec{X} = \vec{b}$$

where A is the matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{1n} \\ a_{12} & a_{22} & a_{23} & a_{2n} \\ & & . & \\ a_{n1} & a_{n2} & a_{n3} & a_{nn} \end{pmatrix}$$

and b is the vector: $\vec{b} = [b_1, b_2, \ldots\ldots, b_n]^T$ Of course, the matrix A and the vector b are known. The goal is to determine the vector X.

2.1 Cramer's Method

This is certainly the most well-known method for solving linear systems. However, we will see that it is the least recommended.

Solution Using Determinants

The system $\vec{A}X = \vec{b}$has a unique solution if and only if the determinant of matrix A is nonzero: det A=|A|≠0, In this case, the solution is given by:

$$x_1 = \frac{|A_1|}{|A|}, \qquad x_2 = \frac{|A_2|}{|A|}, \ldots\ldots, x_n = \frac{|A_n|}{|A|}$$

where A_i is obtained by replacing the i th column of A with the vector b.

Note: If $detA = 0$, the system has either infinitely many solutions or no solution.

Example Solve the following system using Cramer's Method:

$$\begin{cases} x_1 + 3x_2 + 4x_3 = 50 \\ 3x_1 + 5x_2 - 4x_3 = 2 \\ 4x_1 + 7x_2 - 2x_3 = 31 \end{cases}$$

$$\Leftrightarrow \begin{pmatrix} 1 & 3 & 4 \\ 3 & 5 & -4 \\ 4 & 7 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 50 \\ 2 \\ 31 \end{pmatrix}$$

det A=-8≠0⇒A is inversible, In this case, the solution is given by:

$$x_1 = \frac{|A_1|}{|A|} = \frac{|A_1|}{-8},\ A_1 = \begin{pmatrix} 50 & 3 & 4 \\ 2 & 5 & -4 \\ 31 & 7 & -2 \end{pmatrix}, and \quad detA_1 = -21 \Rightarrow x_1 = \frac{|A_1|}{-8} = \frac{-21}{-8} = 3$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{|A_2|}{-8} \quad , A_2 = \begin{pmatrix} 1 & 50 & 4 \\ 3 & 2 & -4 \\ 4 & 31 & -2 \end{pmatrix}, and \quad detA_1 = -40 \Rightarrow x_1 = \frac{|A_1|}{-8} = \frac{-40}{-8} = 5$$

$$x_3 = \frac{|A_3|}{|A|} = , A_2 = \begin{pmatrix} 1 & 3 & 50 \\ 3 & 5 & 2 \\ 4 & 7 & 31 \end{pmatrix}, and \quad detA_1 = -64 \Rightarrow x_1 = \frac{|A_1|}{-8} = \frac{-64}{-8} = 8$$

$$\Leftrightarrow X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix}$$

2.1.1 Solution Using the inverse matrix A^{-1}

If $det\ A = |A| \neq 0 \Rightarrow A^{-1}$ exists.

$$A\vec{X} = \vec{b}$$

$$A^{-1}A\vec{X} = \vec{b}, \Rightarrow \vec{X} = A^{-1}\vec{b} \quad (\quad A^{-1} \text{ is calculated using the method of cofactors}).$$

Remark: Cramer's method requires a large number of computational operations ($(n^2 + n)\, n!$ - 1). If n is large, the number of operations increases and consequently the computation time increases. Moreover, for medium and large systems, the cumulative rounding error increases with the number of operations and affects the accuracy of the results. We will address other methods that require a limited number of calculations, therefore faster and more accurate.

2.3Gaussian Elimination

Introduction (Principle)

This method transforms the system $\vec{A}X = \vec{b}$ into an equivalent system $\overrightarrow{A'}X = \overrightarrow{b'}$where A′ is an upper triangular matrix.

The transformation of the matrix A into A' and the vector b into b' goes through several which we present through the following example:

For better practicality, we form the matrix $\widetilde{A}$ such that $\tilde{A}$=[A⋮b⃗] which is called the augmented matrix of A.

Example : Solve the following system using Gaussian elimination:

$$\begin{cases} x_1 + 3x_2 + 4x_3 = 50 \\ 3x_1 + 5x_2 - 4x_3 = 2 \\ 4x_1 + 7x_2 - 2x_3 = 31 \end{cases}, \quad \tilde{A} = \begin{array}{c} \begin{bmatrix} 1 & 3 & 3 & \vdots & 0 \\ 2 & 2 & 0 & \vdots & 2 \\ 3 & 2 & 6 & \vdots & 11 \end{bmatrix} \end{array} \begin{array}{c} E_1 \\ E_2 \\ E_3 \end{array}$$

Step 1: Eliminate x_{1 }from rows E_2 and E_3

$$E_2 \leftarrow E_2 - \frac{2}{1} E_1, \; E_3 \leftarrow E_3 - \frac{3}{1} E_1$$

Resulting matrix:

$$\widetilde{A^{(1)}} = \begin{bmatrix} 1 & 3 & 3 & \vdots & 0 \\ 0 & -4 & -6 & \vdots & 2 \\ 0 & -7 & -3 & \vdots & 11 \end{bmatrix} \begin{array}{c} E_1 \\ E_2 \\ E_3 \end{array}$$

Step 2: Eliminate $x_2$from E_3: $E_3 \leftarrow E_3 - \left(-\frac{7}{4}\right) E_2$

$$\widetilde{A^{(2)}} = \begin{bmatrix} 1 & 3 & 3 & \vdots & 0 \\ 0 & -4 & -6 & \vdots & 2 \\ 0 & 0 & 15/2 & \vdots & 11 \end{bmatrix} \begin{array}{c} E_1 \\ E_2 \\ E_3 \end{array}$$

Using back-substitution, we find:

15/12 x_3 =15/12 $\Rightarrow$ x_3 =1

-4x_2}-6x_3 =2 $\Rightarrow$$x_2$=-2

x_1+3x_2+3x_3=0$\Rightarrow$$x_1$ }=3

Thus, the solution is:

$$X = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

2.3.1 Algorithm of the Method

$$\begin{cases} \textbf{Triangularization}: \\ \omega = \dfrac{a_{ik}}{a_{kk}} \\ a_{ij} = a_{ij} - \omega a_{kj}; \quad \mathrm{j = k+1, n+1;\ i = k+1, n;\ k = 1, n-1} \\ \textbf{Back} - \textbf{substitution}: \\ x_i = \left(b_i - \displaystyle\sum_{j=i+1}^{n} a_{ij} x_i \right) / a_{ii} \end{cases}$$

2.4 Gaussian Elimination with Pivoting

In Gaussian elimination, we assume at each step that $a_{kk} \neq 0$. However, if a_{kk}=0, we swap rows to ensure a_{kk}≠0

Additionally, dividing by small values can cause errors. To reduce this, we select a_{kk} as the largest absolute value in its column:

$$|a_{kk}| = Max\{|a_{ik}|\}\ ;\ \ k \leq i \leq n$$