Solutions to the Directed Works Series n°6

Exercice n°1 :

We toss a fair coin 3 times. Let X be the number of heads obtained.

1) Find the sample space Ω of this random experiment.

2)) Find the probability distribution of X, its expectation $E(X)$ and its variance $V(X)$.

3) Calculate P (X < 2) and $P(X \geqslant 2)$ and $P(1 < X < 2.5)$.

4)) Let Y = 3X + 1. deduce its expectation E(Y)and its variance V (Y).

Solution :

We toss a fair coin 3 times. Let X be the number of heads obtained.

1. The sample space is :

$$\Omega = \{\text{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}\}, \quad |\Omega| = 8.$$

2. The probability distribution of X is given by :

$$P(X = k) = \frac{|A_k|}{8}, \quad k = 0, 1, 2, 3$$

X	0	1	2	3
P	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Expectation :

$$E(X) = \sum_{k=0}^{3} k\, P(X = k) = \tfrac{3}{2}.$$

Variance :

$$E(X^2) = \sum_{k=0}^{3} k^2\, P(X = k) = 3, \quad V(X) = E(X^2) - [E(X)]^2 = \tfrac{3}{4}.$$

3. Required probabilities :

$$P(X < 2) = P(X = 0) + P(X = 1) = \tfrac{1}{2},$$

$$P(X \geqslant 2) = P(X = 2) + P(X = 3) = \tfrac{1}{2},$$

$$P(1 < X < 2.5) = P(X = 2) = \tfrac{3}{8}.$$

4. For $Y = 3X + 1$:

$$E(Y) = 3E(X) + 1 = \frac{5}{2}, \quad V(Y) = 9V(X) = \frac{27}{4}.$$

Exercice n°2 :

Let X be a random variable whose probability distribution is given by

X	4	5	6	7	8
P	0.15	0.35	0.10	0.25	0.15

1) Find and represent the distribution function of X.

2)) Calculate the following probabilities : $P(X \leq 7.5)$; $P(X > 8)$; $P(4 \leq X \leq 6.5)$; $P(5 < X \leq 6)$.

3) Determine the expectation and variance of X.

4)) Let $Z = \dfrac{1}{X}$. Find the distribution of Z , and its expectation E(Z) and variance and its variance V (Z).

Exercice n°2 : :

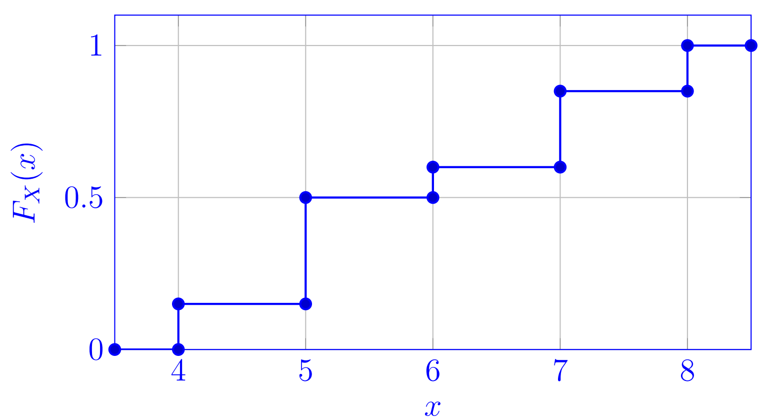
Solution :

Let X be a random variable whose probability distribution is given by :

X	4	5	6	7	8
$P(X)$	0.15	0.35	0.10	0.25	0.15

1. The distribution function $F_X(x) = P(X \leq x)$ is :

$$F_X(x) = \begin{cases} 0, & x < 4, \\ 0.15, & 4 \leq x < 5, \\ 0.50, & 5 \leq x < 6, \\ 0.60, & 6 \leq x < 7, \\ 0.85, & 7 \leq x < 8, \\ 1, & x \geq 8. \end{cases}$$



2. Required probabilities :

$$P(X \leq 7.5) = P(X \leq 7) = 0.85,$$

$$P(X > 8) = 0,$$

$$P(4 \leq X \leq 6.5) = P(X = 4) + P(X = 5) + P(X = 6) = 0.60,$$

$$P(5 < X \leq 6) = P(X = 6) = 0.10.$$

3. Expectation and variance of X :

$$E(X) = 4(0.15) + 5(0.35) + 6(0.10) + 7(0.25) + 8(0.15) = 5.9,$$

$$E(X^2) = 4^2(0.15) + 5^2(0.35) + 6^2(0.10) + 7^2(0.25) + 8^2(0.15) = 36.6,$$

$$V(X) = E(X^2) - [E(X)]^2 = 36.6 - (5.9)^2 = 36.6 - 34.81 = 1.79.$$

4. For $Z = \dfrac{1}{X}$, the distribution is :

Z	$P(Z)$
$1/4$	0.15
$1/5$	0.35
$1/6$	0.10
$1/7$	0.25
$1/8$	0.15

Expectation :

$$E(Z) = 0.17863 \text{ (approx)}.$$

Second moment :

$$E(Z^2) \approx 0.03360.$$

Variance :

$$V(Z) = E(Z^2) - [E(Z)]^2 \approx 0.00170.$$

<u>Exercice n°3 :</u> We consider a biased cubic die whose faces are numbered from 1 to 6, and let X be the random variable given by the number on the top face. We assume that $\exists a \in \mathbb{R}^+$ such that

$$P(X = k) = ka, \quad k = 1, 2, 3, 4, 5, 6.$$

1. Determine the probability distribution of X, and compute its expectation.
2. Let $Y = \dfrac{1}{X}$. Determine the distribution of Y, and compute its expectation and variance.

<u>Solution :</u>

1. **Distribution of X and its expectation**

   Normalization gives :

$$\sum_{k=1}^{6} \mathbb{P}(X=k) = \sum_{k=1}^{6} ka = a\sum_{k=1}^{6} k = a \cdot 21 = 1 \quad \Rightarrow \quad a = \frac{1}{21}.$$

   Hence,

$$\mathbb{P}(X=k) = \frac{k}{21}, \quad k = 1, \dots, 6.$$

k	$\mathbb{P}(X=k)$
1	$\frac{1}{21}$
2	$\frac{2}{21}$
3	$\frac{3}{21}$
4	$\frac{4}{21}$
5	$\frac{5}{21}$
6	$\frac{6}{21}$

   Expectation :

$$E(X) = \sum_{k=1}^{6} k \cdot \frac{k}{21} = \frac{1^2+2^2+3^2+4^2+5^2+6^2}{21} = \frac{91}{21} = \frac{13}{3}.$$

   Second moment :

$$E(X^2) = \sum_{k=1}^{6} k^2 \cdot \frac{k}{21} = \frac{\sum_{k=1}^{6} k^3}{21} = \frac{(1+2+\cdots+6)^2}{21} = \frac{21^2}{21} = 21.$$

   Variance :

$$V(X) = E(X^2) - [E(X)]^2 = 21 - \left(\frac{13}{3}\right)^2 = \frac{20}{9}.$$

2. **Distribution of $Y = \dfrac{1}{X}$, expectation, and variance**

   Support :

$$Y \in \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}\right\}, \quad \mathbb{P}\left(Y = \frac{1}{k}\right) = \mathbb{P}(X=k) = \frac{k}{21}, \quad k = 1, \dots, 6.$$

   Expectation :

$$E(Y) = \sum_{k=1}^{6} \frac{1}{k} \cdot \frac{k}{21} = \sum_{k=1}^{6} \frac{1}{21} = \frac{6}{21} = \frac{2}{7}.$$

   Second moment :

$$E(Y^2) = \sum_{k=1}^{6} \frac{1}{k^2} \cdot \frac{k}{21} = \frac{1}{21}\sum_{k=1}^{6} \frac{1}{k} = \frac{H_6}{21} = \frac{\frac{147}{60}}{21} = \frac{49}{420} = \frac{7}{60}.$$

   Variance :

$$V(Y) = E(Y^2) - [E(Y)]^2 = \frac{7}{60} - \left(\frac{2}{7}\right)^2 = \frac{7}{60} - \frac{4}{49} = \frac{103}{2940}.$$

Exercice n°4 :

Let X be a random variable with The probability density function (PDF) given by

$$f(x) = \begin{cases} kx^2, & \text{if } -1 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

1. Find the constant k.

2. Calculate $E(X)$ and $Var(X)$.
3. Determine $F(x)$.
4. Calculate $P_{0\leq X\leq 2}(X \geq 1)$.

Solution :

Let X be a random variable with the probability density function

$$f(x) = \begin{cases} kx^2, & -1 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

1. **Find the constant k :**

$$\int_{-1}^{2} kx^2\,dx = 1 \quad \Rightarrow \quad k \cdot \left[\frac{x^3}{3}\right]_{-1}^{2} = k \cdot 3 = 1 \quad \Rightarrow \quad k = \frac{1}{3}.$$

2. **Calculate $E(X)$ and $Var(X)$:**

$$E(X) = \int_{-1}^{2} xf(x)\,dx = \frac{1}{3}\int_{-1}^{2} x^3\,dx = \frac{1}{3}\left[\frac{x^4}{4}\right]_{-1}^{2} = \frac{5}{4}.$$

$$E(X^2) = \int_{-1}^{2} x^2 f(x)\,dx = \frac{1}{3}\int_{-1}^{2} x^4\,dx = \frac{11}{5}.$$

$$Var(X) = E(X^2) - (E(X))^2 = \frac{11}{5} - \left(\frac{5}{4}\right)^2 = \frac{51}{80}.$$

3. **Determine $F(x)$:**

$$\begin{aligned} F(x) &= \mathrm{P}(X \leq x) \\ &= \int_{-\infty}^{x} f(t)dt \quad . \end{aligned}$$

Then,

$$F(x) = \begin{cases} 0, \text{ if } x < -1 \\ \int_{-1}^{x} \frac{1}{3}t^2 dt = \frac{x^3+1}{9} \text{ if } -1 \leq x < 2 \\ 1, \text{ if } x \geq 2 \end{cases}$$

4. **Calculate $P_{0\leq X\leq 2}(X \geq 1)$:**

$$\begin{aligned} P_{(0\leq X\leq 2)}(1 \leq X) &= \frac{P(\{1 \leq X\} \cap \{0 \leq X \leq 2\})}{P(\{0 \leq X \leq 2\})} \\ &= \frac{P(\{1 \leq X \leq 2\})}{P(\{0 \leq X \leq 2\})} \end{aligned}$$

$$P(1 \leq X \leq 2) = \int_{1}^{2} \tfrac{1}{3}x^2\,dx = \tfrac{7}{9}, \quad P(0 \leq X \leq 2) = \int_{0}^{2} \tfrac{1}{3}x^2\,dx = \tfrac{8}{9}.$$

$$\Rightarrow \quad P(X \geq 1 \mid 0 \leq X \leq 2) = \frac{7}{8}.$$

Exercice n°5 :

The cumulative distribution function of a continuous r.v. Y is given by

$$F(y) = \begin{cases} 1 - \dfrac{9}{y^2}, & \text{if } 3 < y \\ 0, & \text{otherwise} \end{cases}$$

1. Find $P(Y \leq 5)$, and $P(Y > 8)$.
2. The pdf of Y
3. Calculate $E(X)$ and $Var(X)$.

Solution :

The cumulative distribution function of a continuous r.v. Y is given by

$$F(y) = \begin{cases} 1 - \dfrac{9}{y^2}, & \text{if } 3 < y \\ 0, & \text{otherwise} \end{cases}$$

1. **Probabilities :**

$$P(Y \leq 5) = F(5) = 1 - \frac{9}{25} = \frac{16}{25},$$

$$P(Y > 8) = 1 - F(8) = 1 - \left(1 - \frac{9}{64}\right) = \frac{9}{64}.$$

2. **PDF of Y :**

$$f(y) = F'(y) = \frac{d}{dy}\left(1 - \frac{9}{y^2}\right) = \frac{18}{y^3}, \quad y > 3,$$

and $f(y) = 0$ otherwise.

3. **Expectation and variance :**

$$E(Y) = \int_3^{\infty} y f(y)\, dy = \int_3^{\infty} y \cdot \frac{18}{y^3}\, dy = \int_3^{\infty} \frac{18}{y^2}\, dy = 18\left[-\frac{1}{y}\right]_3^{\infty} = 6.$$

$$E(Y^2) = \int_3^{\infty} y^2 f(y)\, dy = \int_3^{\infty} y^2 \cdot \frac{18}{y^3}\, dy = \int_3^{\infty} \frac{18}{y}\, dy = \infty.$$

So $Var(Y)$ does not exist (it is infinite)