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Faculty of Technology
Department of Civil Engineering-Department of Electrical Engineering
Module: Probability-Statistics

Chapter 7: Common discrete probability distributions

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December 8, 2025

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1 Introduction

2 Discrete Uniform Distribution

Definition 1. *A random variable X follows a* ***discrete uniform distribution*** *on the set $\{x_1, x_2, \ldots, x_n\}$ if:*

*

1. *The image of the sample space is* $X(\Omega) = \{x_1, x_2, \ldots, x_n\}$.
2. *For all* $k \in \{x_1, x_2, \ldots, x_n\}$,

$$P(X = k) = \frac{1}{n}.$$

We denote this as $X \sim U\{x_1, \ldots, x_n\}$.

Proposition 1 (Expectation and Variance)**.**

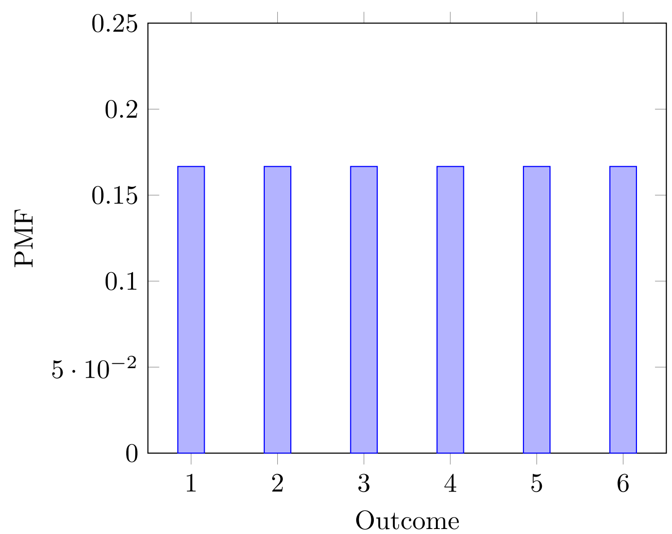
$$E[X] = \frac{1}{n}\sum_{i=1}^{n} x_i, \quad Var(X) = \frac{1}{n}\sum_{i=1}^{n}(x_i - E[X])^2.$$

Proof 1. *Since all values have equal probability* $1/n$, $E[X]$ *is the arithmetic mean of the support. Variance follows directly from the definition* $Var(X) = E[(X - E[X])^2]$ *with uniform weights* $1/n$.

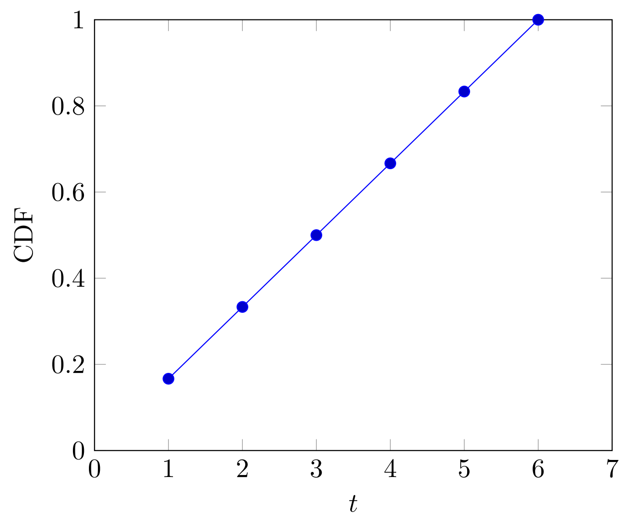
Exercice 1. *Let* $X \sim U\{1, 2, 3, 4, 5, 6\}$ *(fair die). Compute* $P(X \leq 3)$.

$$P(X \leq 3) = \frac{3}{6} = \frac{1}{2}.$$

PMF (die):



CDF (die):



3 Bernoulli Distribution

Definition 2. *A random variable X follows a **Bernoulli distribution** with parameter p if:*

1. *The image of the sample space is $X(\Omega) = \{0, 1\}$.*
2. *For all $x \in \{0, 1\}$,*

$$P(X = 1) = p, \quad P(X = 0) = 1 - p.$$

We denote this as $X \sim B(p)$.

Proposition 2 (Expectation and Variance)**.**

$$E[X] = p, \quad Var(X) = p(1 - p).$$

Proof 2.

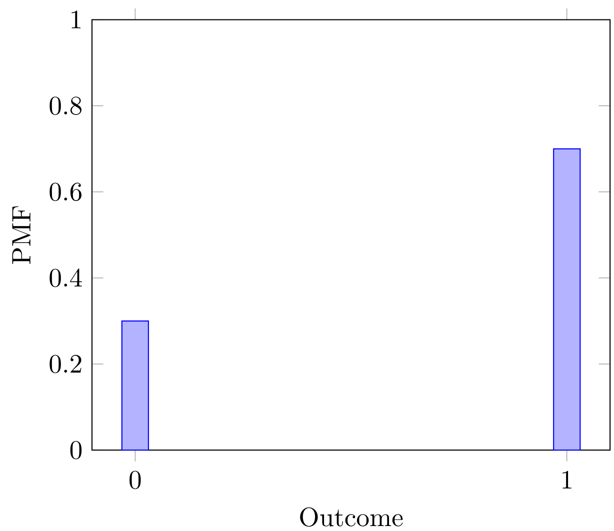
$$E[X] = 0 \cdot (1 - p) + 1 \cdot p = p, \quad E[X^2] = 0^2(1 - p) + 1^2 p = p.$$

Thus $Var(X) = E[X^2] - (E[X])^2 = p - p^2 = p(1 - p)$.

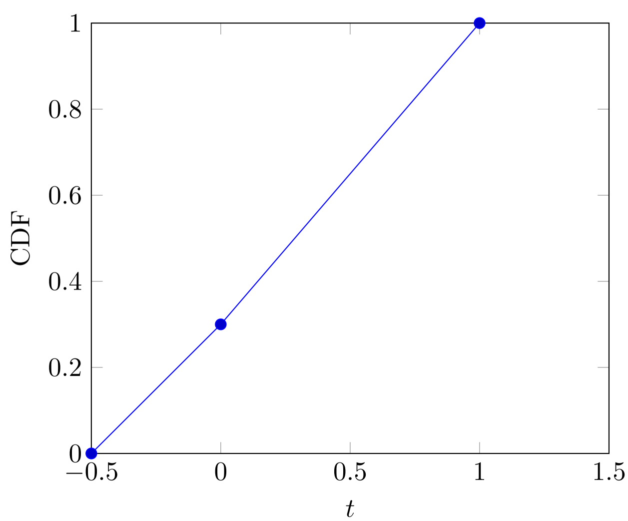
Exercice 2. *Let $X \sim B(0.7)$. Compute $P(X = 0)$ and $Var(X)$.*

$$P(X = 0) = 0.3, \quad Var(X) = 0.7 \cdot 0.3 = 0.21.$$

PMF (p=0.7):



CDF (p=0.7):



4 Binomial Distribution

Definition 3. *A random variable X follows a* ***Binomial distribution*** *with parameters n, p if:*

1. $X(\Omega) = \{0, 1, \ldots, n\}$.
2. $\forall\ k \in \{0, 1, \ldots, n\}$,

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

We denote this as $X \sim Bin(n,p)$.

Proposition 3. *For integers $n \geq 1$ and $1 \leq k \leq n$, we have*

$$n\binom{n-1}{k-1} = k\binom{n}{k}.$$

Proof. Recall the definition of the binomial coefficient:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Compute the left-hand side:

$$n\binom{n-1}{k-1} = n \cdot \frac{(n-1)!}{(k-1)!(n-k)!} = \frac{n!}{(k-1)!(n-k)!}.$$

Compute the right-hand side:

$$k\binom{n}{k} = k \cdot \frac{n!}{k!(n-k)!} = \frac{n!}{(k-1)!(n-k)!}.$$

Thus both sides are equal:

$$n\binom{n-1}{k-1} = k\binom{n}{k}.$$

□

Proposition 4 (Expectation and Variance)**.**
Let $X \sim \mathrm{Bin}(n,p)$ with $q = 1-p$. Then

$$E[X] = np \quad \textit{and} \quad \mathrm{Var}(X) = npq.$$

Proof. We start from the definition

$$E[X] \;=\; \sum_{k=0}^{n} k\binom{n}{k} p^k q^{\,n-k}.$$

Since the term for $k=0$ is zero, we can write

$$E[X] \;=\; \sum_{k=1}^{n} k\binom{n}{k} p^k q^{\,n-k}.$$

Use the identity $k\binom{n}{k} = n\binom{n-1}{k-1}$, and factor out np:

$$E[X] \;=\; np\sum_{k=1}^{n} \binom{n-1}{k-1} p^{\,k-1} q^{\,(n-1)-(k-1)}.$$

Let $j = k - 1$. Then $j = 0, 1, \ldots, n-1$, and

$$E[X] \;=\; np \sum_{j=0}^{n-1} \binom{n-1}{j} p^{j} q^{(n-1)-j} \;=\; np\,(p+q)^{n-1} \;=\; np.$$

For the variance, we use $\mathrm{Var}(X) = E[X^2] - \big(E[X]\big)^2$. Compute $E[X^2]$ as

$$E[X^2] \;=\; \sum_{k=0}^{n} k^2 \binom{n}{k} p^k q^{n-k}.$$

Note that $k^2 = k(k-1) + k$, hence

$$E[X^2] \;=\; \sum_{k=0}^{n} k(k-1) \binom{n}{k} p^k q^{n-k} \;+\; \sum_{k=0}^{n} k \binom{n}{k} p^k q^{n-k}.$$

For the first sum, use $k(k-1)\binom{n}{k} = n(n-1)\binom{n-1}{k-1}$ and factor $n(n-1)p^2$:

$$\sum_{k=0}^{n} k(k-1)\binom{n}{k} p^k q^{n-k} \;=\; n(n-1)p^2 \sum_{k=2}^{n} \binom{n-2}{k-2} p^{k-2} q^{(n-2)-(k-2)} \;=\; n(n-1)p^2 (p+q)^{n-2} \;=\; n(n-1)p^2.$$

For the second sum we already have $E[X] = np$. Therefore

$$E[X^2] \;=\; n(n-1)p^2 \;+\; np.$$

Finally,

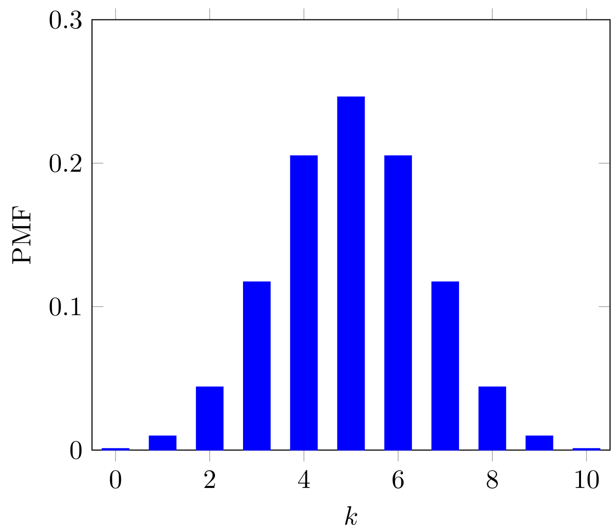
$$\mathrm{Var}(X) \;=\; E[X^2] - \big(E[X]\big)^2 \;=\; \big(n(n-1)p^2 + np\big) - n^2p^2 \;=\; np - np^2 \;=\; np(1-p) \;=\; npq.$$

$\square$

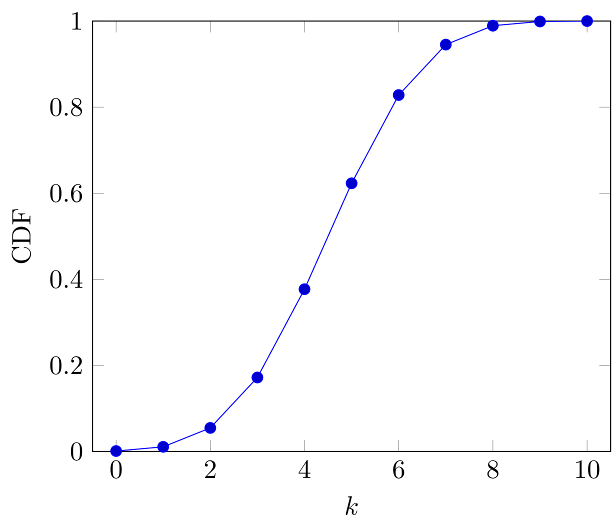
Exercice 3. *Let $X \sim Bin(10, 0.5)$. Compute $P(X = 5)$.*

$$P(X=5) = \binom{10}{5}(0.5)^{10} = \frac{252}{1024} \approx 0.24609375.$$

PMF (n=10, p=0.5):



CDF (n=10, p=0.5):



5 Poisson Distribution

Definition 4. *A random variable X follows a* ***Poisson distribution*** *with parameter $\lambda > 0$ if:*

1. *The image of the sample space is $X(\Omega) = \{0, 1, 2, \dots\}$.*
2. *For all $k \in \{0, 1, 2, \dots\}$,*

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}.$$

We denote this as $X \sim Poisson(\lambda)$.

Proposition 5 (Expectation and Variance)**.**
Let $X \sim \text{Poisson}(\lambda)$, $\lambda > 0$. *Then*

$$E[X] = \lambda \quad \textit{and} \quad \text{Var}(X) = \lambda.$$

Proof. By definition of the Poisson pmf, for $n = 0, 1, 2, \ldots$,

$$P(X = n) = e^{-\lambda}\frac{\lambda^n}{n!}.$$

Expectation.

$$E[X] \;=\; \sum_{n=0}^{\infty} n\, e^{-\lambda}\frac{\lambda^n}{n!} \;=\; e^{-\lambda}\sum_{n=1}^{\infty} n\frac{\lambda^n}{n!} \;=\; e^{-\lambda}\sum_{n=1}^{\infty}\frac{\lambda^n}{(n-1)!}.$$

Let $m = n - 1$. Then

$$E[X] \;=\; e^{-\lambda}\sum_{m=0}^{\infty}\frac{\lambda^{m+1}}{m!} \;=\; \lambda e^{-\lambda}\sum_{m=0}^{\infty}\frac{\lambda^m}{m!} \;=\; \lambda e^{-\lambda}e^{\lambda} \;=\; \lambda.$$

Second factorial moment. Consider

$$E\big[X(X-1)\big] \;=\; \sum_{n=0}^{\infty} n(n-1)\, e^{-\lambda}\frac{\lambda^n}{n!} \;=\; e^{-\lambda}\sum_{n=2}^{\infty} n(n-1)\frac{\lambda^n}{n!}.$$

Note that $n(n-1)\dfrac{\lambda^n}{n!} = \dfrac{\lambda^n}{(n-2)!}$. Hence

$$E\big[X(X-1)\big] \;=\; e^{-\lambda}\sum_{n=2}^{\infty}\frac{\lambda^n}{(n-2)!} \;=\; e^{-\lambda}\sum_{m=0}^{\infty}\frac{\lambda^{m+2}}{m!} \;=\; \lambda^2 e^{-\lambda}\sum_{m=0}^{\infty}\frac{\lambda^m}{m!} \;=\; \lambda^2.$$

Variance. Since

$$E\big[X(X-1)\big] \;=\; E[X^2] - E[X],$$

we obtain

$$E[X^2] \;=\; \lambda^2 + \lambda.$$

Therefore,

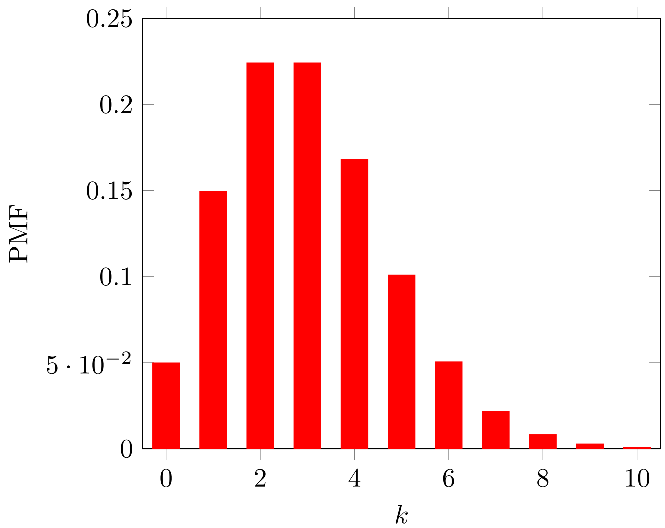
$$\text{Var}(X) \;=\; E[X^2] - \big(E[X]\big)^2 \;=\; (\lambda^2 + \lambda) - \lambda^2 \;=\; \lambda.$$

□

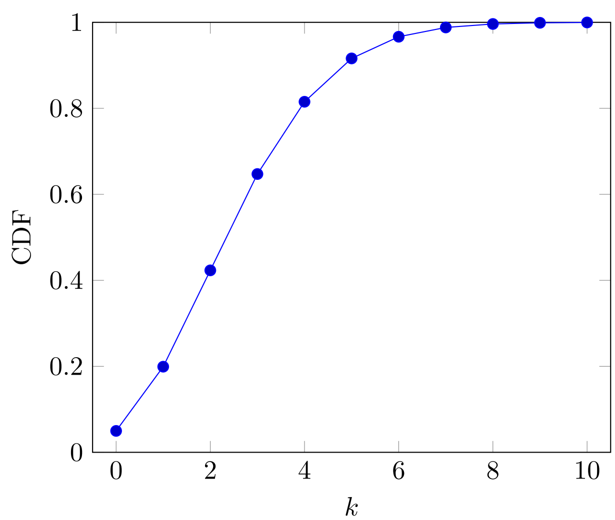
Exercice 4. *Let $X \sim Poisson(3)$. Compute $P(X = 0)$ and $P(X = 1)$.*

$$P(X = 0) = e^{-3} \approx 0.049787, \quad P(X = 1) = 3e^{-3} \approx 0.149361.$$

PMF ($\lambda = 3$):



CDF ($\lambda = 3$, cumulative up to k):



6 Geometric Distribution

Definition 5. *A random variable X follows a* ***Geometric distribution*** *with parameter $p \in (0, 1]$ if:*

1. *The image of the sample space is* $X(\Omega) = \{1, 2, \dots\}$.
2. *For all* $k \in \{1, 2, \dots\}$,

$$P(X = k) = (1 - p)^{k-1}p.$$

We denote this as $X \sim Geom(p)$.

Proposition 6 (Expectation and Variance)**.** *Let* $X \sim \mathrm{Geom}(p)$*, with parameter* $p \in (0, 1]$*. Then*

$$E[X] = \frac{1}{p}, \quad \mathrm{Var}(X) = \frac{1 - p}{p^2}.$$

Proof. By definition, for $k = 1, 2, \dots$,

$$P(X = k) = (1 - p)^{k-1}p.$$

Expectation.

$$E[X] = \sum_{k=1}^{\infty} k(1 - p)^{k-1}p.$$

Factor p:

$$E[X] = p\sum_{k=1}^{\infty} k(1 - p)^{k-1}.$$

Recall the series identity

$$\sum_{k=1}^{\infty} kr^{k-1} = \frac{1}{(1 - r)^2}, \quad |r| < 1.$$

Here $r = 1 - p$. Thus

$$E[X] = p \cdot \frac{1}{p^2} = \frac{1}{p}.$$

Variance.

$$E[X^2] = \sum_{k=1}^{\infty} k^2(1 - p)^{k-1}p.$$

Write $k^2 = k(k - 1) + k$. Then

$$E[X^2] = p\sum_{k=1}^{\infty} k(k - 1)(1 - p)^{k-1} + p\sum_{k=1}^{\infty} k(1 - p)^{k-1}.$$

For the second sum we already know it equals $1/p$. For the first sum, let's compute:

$$\sum_{k=1}^{\infty} k(k-1)(1-p)^{k-1} = (1-p)\sum_{k=2}^{\infty} k(k-1)(1-p)^{k-2}.$$

Set $j = k-2$. Then

$$\sum_{k=2}^{\infty} k(k-1)(1-p)^{k-2} = \sum_{j=0}^{\infty}(j+2)(j+1)(1-p)^{j}.$$

But a simpler way: use the known identity

$$\sum_{k=1}^{\infty} k(k-1)r^{k-1} = \frac{2r}{(1-r)^3}, \quad |r| < 1.$$

Here $r = 1-p$. So

$$\sum_{k=1}^{\infty} k(k-1)(1-p)^{k-1} = \frac{2(1-p)}{p^3}.$$

Therefore

$$E[X^2] = p \cdot \frac{2(1-p)}{p^3} + p \cdot \frac{1}{p^2} = \frac{2(1-p)}{p^2} + \frac{1}{p}.$$

Variance.

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2.$$

Substitute:

$$\mathrm{Var}(X) = \left(\frac{2(1-p)}{p^2} + \frac{1}{p}\right) - \left(\frac{1}{p}\right)^2.$$

Simplify:

$$\mathrm{Var}(X) = \frac{2(1-p)}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{2(1-p)}{p^2} + \frac{p}{p^2} - \frac{1}{p^2} = \frac{2(1-p)+p-1}{p^2} = \frac{1-p}{p^2}.$$

Thus

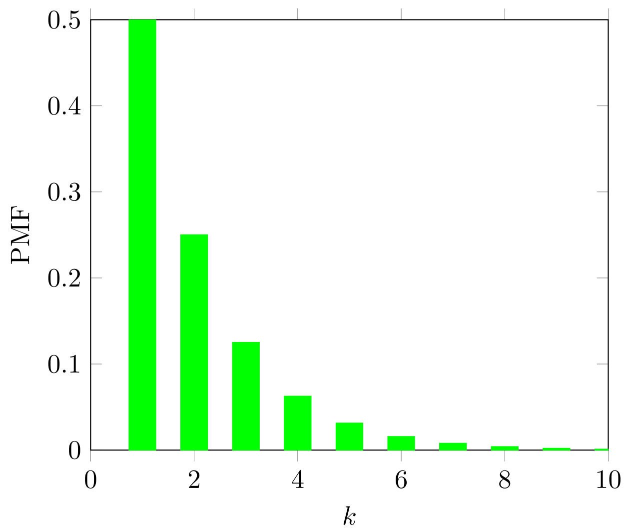
$$E[X] = \frac{1}{p}, \quad \mathrm{Var}(X) = \frac{1-p}{p^2}.$$

$\square$

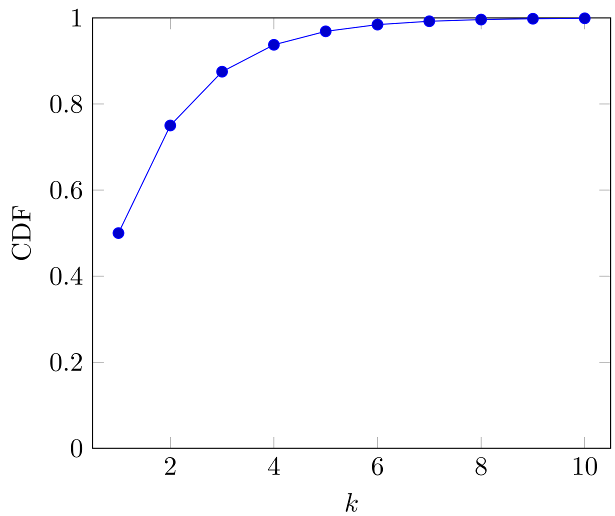
Exercice 5. *Let $X \sim Geom(0.5)$. Compute $P(X \leq 3)$.*

$$P(X \leq 3) = 1-(1-p)^3 = 1-(0.5)^3 = 1-0.125 = 0.875.$$

PMF (p=0.5):



CDF (p=0.5):



Properties of Common Probability Distributions

Distribution	**Mean**	**Variance**	**Support**
Bernoulli(p)	p	$p(1-p)$	$x \in \{0, 1\}$
Binomial(n, p)	np	$np(1-p)$	$x = 0, 1, \dots, n$
Poisson(λ)	λ	λ	$x = 0, 1, 2, \dots$
Geometric(p)	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$x = 1, 2, \dots$
Uniform(a, b)	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$x \in [a, b]$