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Module: Probability-Statistics
Chapter 6: Random variables

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1 Introduction

The notion of **random variable** plays an important role in probability. The majority of experiments can be modeled simply using a random variable.

2 Definitions and properties

Definition 1. A **random variable** X , defined on a sample space Ω , is a function from Ω to the set of real numbers $\mathbb{R}$, where every possible outcome of the experiment (i.e., every element of Ω) is associated with a real number

$$\begin{aligned} X : \Omega &\longrightarrow \mathbb{R} \\ \omega &\longmapsto X(\omega) \end{aligned}$$

Remarks

- We use a capital letter, say X , to denote a random variable, and use the corresponding lower-case letter x to denote the realization (values) of the random variable.
- In general, the set of values taken by the random variable X is denoted by $X(\Omega)$ and is called the **range** of the random variable X .

• Let X and Y be two random variables defined on a sample space (Ω) , then $X + Y$, XY , $X + \lambda$, and $\lambda.X$ (where λ is any real number) $upper(X; Y)$ and $lower(X; Y)$ are also random variables on Ω .

- If $X \neq 0$, then $\frac{1}{X}$ is random variable on Ω .

Example 1. We toss a coin 2 times and observe the sequence of heads/tails. The sample space here may be defined as

$$\Omega = \{(H, H), (H, T), (T, H), (T, T)\}.$$

Let X represent the number of heads that can come up.

$$X(T, T) = 0; X(H, T) = 1; X(T, H) = 1, X(H, H) = 2.$$

So the values that the random variable X can take are: 0; 1; 2. i.e.

$$X(\Omega) = \{0, 1, 2\}$$

Example 2. A box contains three balls numbered 2, 3 and 5. We draw successively with return $\Omega = \{(2; 2); (2; 3); (2; 5); (3; 2); (3; 3); (3; 5); (5; 2); (5; 3); (5; 5)\}$ Let Y be the random variable indicating the sum of the points obtained

$$Y : \Omega \longrightarrow \mathbb{R}$$

$$(i, j) \mapsto Y(i, j) = i + j$$

Therefore, $Y(\Omega) = \{4; 5; 6; 7; 8; 10\}$.

Remark 1. There are two types of random variables:

1. Discrete random variables ;
2. Continuous random variables.

3 Discrete Random Variable

Definition 2. A random variable X is said to be discrete random variable if its **range** $X(\Omega)$ includes finite number of values or countable infinite number of values.

Example 3. The random variables defined in the previous examples are discrete. X can only take three values, 0, 1 or 2. Y can only take six values, 4; 5; 6; 7; 8 or 10.

3.1 Discrete Probability Distributions

The probability distribution of X is usually given in the form of the following table following table

X	x_1	x_2	$\dots$	x_n
P	p_1	p_2	$\dots$	p_n

The probability distribution satisfies the conditions :

1. $0 \leq P(X = x_i) \leq 1$.
2. $\sum_{i=1}^n P(X = x_i) = 1$.

Example 4. In example 1, the probability distribution of **boldsymbol** X is

X	0	1	2
P	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

Example 5. Taking example 2 again, the probability distribution of Y is

Y	4	5	6	7	8	10
P	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{1}{9}$

3.2 Graphical representation of the probability distribution

This is done using a bar chart with the values taken by the random variable on the abscissa and the corresponding probability values on the ordinate.

Example 6. *In the example of the coin toss: (example 1)*

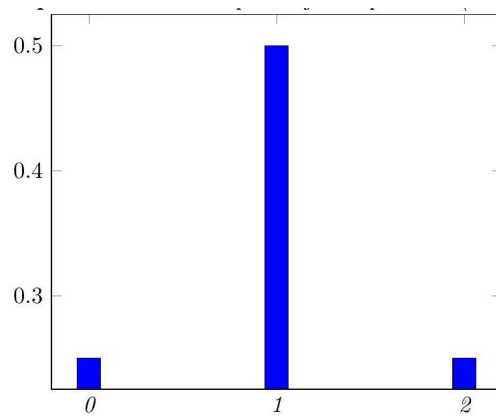


Figure 1: Distribution of X

Example 7. *Let's take the random variable Y indicating the sum of points obtained after two draws with a discount (example 2).*

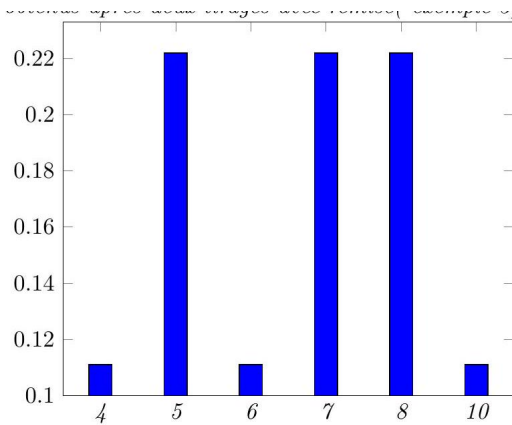


Figure 2: Distribution of Y

3.3 Cumulative distribution function for Discrete Random Variables

Definition 3. Let X be a discrete random variable on Ω . The cumulative distribution function of X is the function F_X defined by :

$$F_X : \mathbb{R} \rightarrow [0; 1]$$

$$x \mapsto F_X(x) = P(X \leq x) = \sum_{x_i \leq x} P(X = x_i)$$

If X takes on only a finite number of values $x_1, x_2, \dots, x_n$, then the distribution function is given by

$$F_X = \begin{cases} 0, & x < x_1 \\ P(X = x_1), & x_1 \leq x < x_2 \\ P(X = x_1) + P(X = x_2), & x_2 \leq x < x_3 \\ \vdots & \\ P(X = x_1) + \dots + P(X = x_n), & x_n \leq x \end{cases}$$

3.3.1 Properties

The distribution function $F(x)$ has the following properties:

- 1) $F(x)$ is nondecreasing [i.e., $F(x) \leq F(y)$ if $x \leq y$]
- 2) $\lim_{x \rightarrow -\infty} F(x) = 0, \quad \lim_{x \rightarrow +\infty} F(x) = 1.$
- 3) $F(x)$ is continuous from the right.

Example 8. (a) Find the distribution function for the random variable X of Example 1.

(b) Obtain its graph.

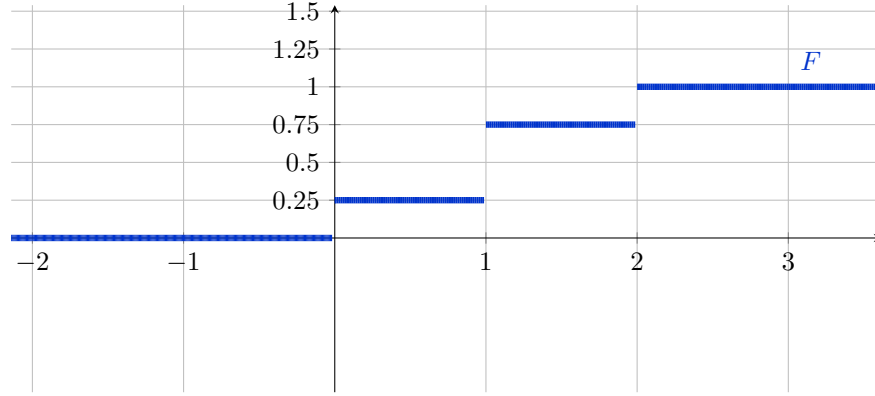
Answer

(a) The distribution function is

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{4}, & 0 \leq x < 1 \\ \frac{3}{4}, & 1 \leq x < 2 \\ 1, & 2 \leq x. \end{cases}$$

(b) The graph of $F(x)$ is shown in Figure

Section 334 Relationship between distribution function and probability of X



Exercise 1. (a) Find the distribution function for the random variable Y of Example 2.

(b) Obtain its graph.

Answer See worksheet.

3.4 Relationship between distribution function and probability of X

Theorem 1. Let $a, b \in \mathbb{R}$ with $a < b$, X be a discrete random variable and F_X its function then :

- 1) $P(a < X \leq b) = F_X(b) - F_X(a)$.
- 2) $P(a \leq X \leq b) = F_X(b) - F_X(a) + P(X = a)$.
- 3) $P(a < X < b) = F_X(b) - F_X(a) - P(X = b)$.
- 4) $P(a \leq X < b) = F_X(b) - F_X(a) + P(X = a) - P(X = b)$.
- 5) $P(a < X) = 1 - F_X(a)$.
- 6) $P(a \leq X) = 1 - F_X(a) + P(X = a)$.

Proof. Clearly it is sufficient to show the first equality, let $a, b \in \mathbb{R}$ with $a < b$, we have :

$\{X \leq b\} = \{a < X \leq b\} \cup \{X \leq a\}$ and $\phi = \{a < X \leq b\} \cap \{X \leq a\}$
according to Axiom 3 (Chapter 4, page 6)

$$P(X \leq b) = P(\{a < X \leq b\} \cup \{X \leq a\}) = P(\{a < X \leq b\}) + P(\{X \leq a\})$$

$$F_X(b) = P(a < X \leq b) + F_X(a)$$

Therefore

$$P(a < X \leq b) = F_X(b) - F_X(a)$$

□

Example 9. In example 2, calculate the following probabilities:

$$P(Y \leq 8) = F(8) = 8/9.$$

$$P(Y > 5) = 1 - P(Y \leq 5) = 1 - F(5) = 1 - 3/9 = 6/9$$

$$P(6 \leq Y \leq 7.5) = P(6 \leq Y \leq 7) = F(7) - F(6) + P(6) = 6/9 - 4/9 + 1/9 = 3/9.$$

3.5 Mathematical Expectation

Definition 3. For a discrete random variable X having the possible values $x_1, \dots, x_n$, with respective probabilities $P(X = x_1); P(X = x_n)$: The **mathematical expectation** of X is :

$$m = E(X) = \sum_{i=1}^n x_i P(X = x_i)$$

Example 10. In example 1, we have

$$m = E(X) = \sum_{i=1}^n x_i P(X = x_i) = 0 \times \frac{1}{4} + 1 \times \frac{2}{4} + 2 \times \frac{1}{4} = 1$$

Example 11. In example 2, we have

$$m = E(X) = \sum_{i=1}^n x_i P(X = x_i) = 7$$

3.5.1 Interpretation

The expectation $E[X]$ represents the mean value taken by X .

3.5.2 Properties of mathematical expectation

Let X and Y be two random variables defined on Ω , a and b two real constants.

1. $E(\alpha X) = \alpha E(X)$
2. $E(E(X)) = E(X)$; since $E(X)$ is not a random variable.
3. $E(aX + bY) = aE(X) + bE(Y)$

3.6 Variance and standard deviation of a discrete random variable

Definition 4. The variance is the quantity:

$$V(x) = \sum_{i=1}^n P(X = x_i)(x_i - m)^2 \quad (1)$$

Definition 5. The standard deviation is the quantity

$$\sigma_x = \sqrt{V(x)} \quad (2)$$

Proposition 1. The variance of X , can be calculated as $m = E(X)$:

$$V(x) = \sum_{i=1}^n P(X = x_i)x_i^2 - m^2 \quad (3)$$

Proof. exercice □

3.6.1 Properties of $V(X)$

Let X be a random variable defined on Ω , with α and β two real constants.

- 1) The variance of a constant is zero: $V(\alpha) = 0$
- 2) $V(X + \alpha) = V(X)$.
- 3) $V(\beta X + \alpha) = \beta^2 V(X)$

4 Continuous random variable

Definition 6. A random variable X is said to be **continuous** when it can take on an infinite, non-countable number of values. This means that such a variable can take all the values of a real interval.

Example 12. Examples of continuous random variables include:

- The weight of a child at birth (e.g., between 2.7kg and 5.6kg).
- The time interval between two train passes.
- The lifetime in seconds of a mechanical part.

4.1 Probability Density Function (PDF)

The function $f(x)$ is a **probability density function (PDF)** for the continuous random variable X , defined over the set of real numbers, if:

1. $f(x) \geq 0 \quad \forall x \in \mathbb{R}$
2. $\int_{-\infty}^{\infty} f(x) dx = 1$
3. $P(a < X < b) = \int_a^b f(x) dx$

Note. If X is a continuous random variable, then:

$$\begin{aligned} P(a < X < b) &= P(a \leq X < b) \\ &= P(a < X \leq b) \\ &= P(a \leq X \leq b) \end{aligned}$$

This equality holds because the probability of X taking any exact value is zero in the continuous case. This is **not** the case for discrete random variables.

4.2 Cumulative Distribution Function (CDF)

The **cumulative distribution function (CDF)** of X is defined as:

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(t) dt$$

Properties of CDF

- $F(x)$ is non-decreasing.
- $\lim_{x \rightarrow -\infty} F(x) = 0$
- $\lim_{x \rightarrow +\infty} F(x) = 1$
- If $F(x)$ is differentiable, then:

$$f(x) = \frac{d}{dx} F(x)$$

4.3 Probability of an Interval

For a continuous random variable X , the probability that X belongs to an interval $[x_1, x_2]$ can be obtained either by integrating the PDF or by using the CDF:

$$P(X \in [x_1, x_2]) = \int_{x_1}^{x_2} f(t) dt = F(x_2) - F(x_1)$$

Example 13. Consider continuous random variable X , with PDF:

$$f(x) = e^{-x}, \quad x \geq 0$$

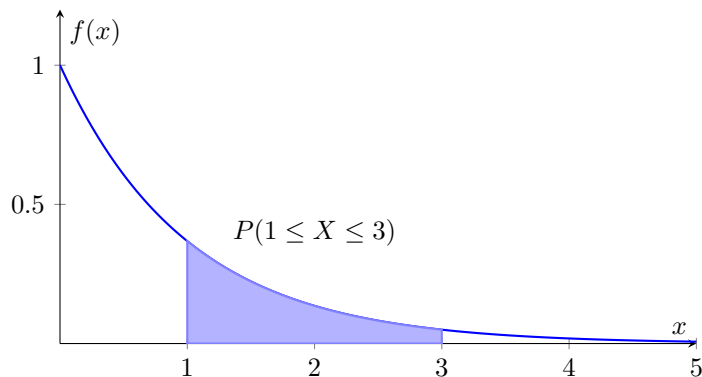
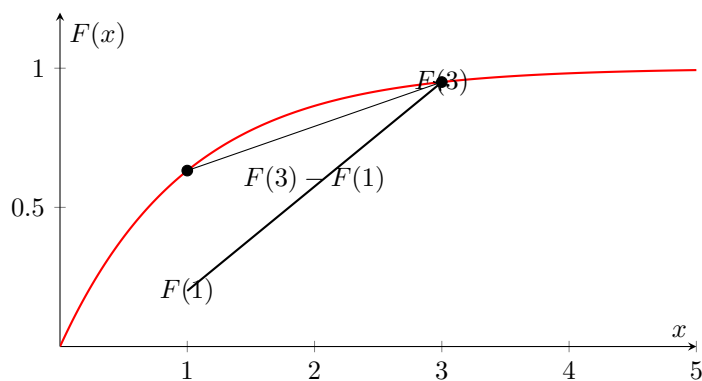
The CDF is:

$$F(x) = 1 - e^{-x}, \quad x \geq 0$$

Probability on Interval $[1, 3]$

$$\begin{aligned} P(1 \leq X \leq 3) &= \int_1^3 e^{-t} dt = e^{-1} - e^{-3} \\ &= F(3) - F(1) = (1 - e^{-3}) - (1 - e^{-1}) = e^{-1} - e^{-3} \end{aligned}$$

Graphical Illustration

PDF of $\text{Exp}(1)$ with shaded probabilityCDF of $\text{Exp}(1)$ showing $F(3) - F(1)$ 

Example 14. Consider continuous random variable X , with PDF:

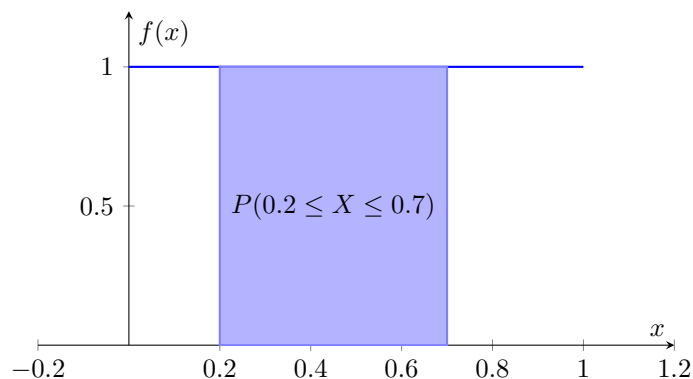
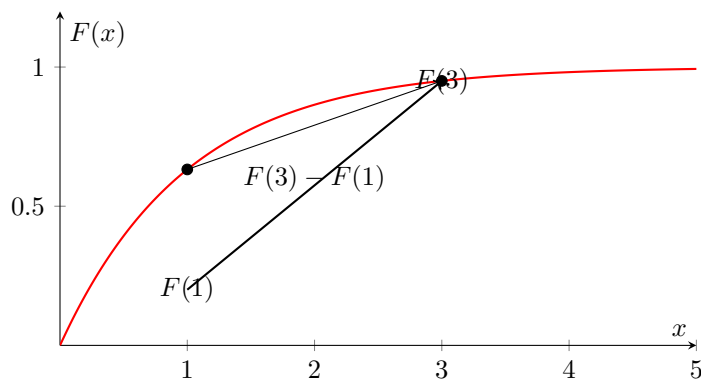
$$f(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

The CDF is:

$$F(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

Probability on Interval $[0.2, 0.7]$

$$\begin{aligned} P(0.2 \leq X \leq 0.7) &= \int_{0.2}^{0.7} 1 \, dx = 0.5 \\ &= F(0.7) - F(0.2) = 0.7 - 0.2 = 0.5 \end{aligned}$$

Graphical IllustrationPDF of $U(0, 1)$ with shaded probabilityCDF of $Exp(1)$ showing $F(3) - F(1)$ **4.4 Moments of a Continuous Random Variable****4.4.1 Mathematical Expectation (Mean)**

The **expected value** or **mean** of a continuous random variable X with probability density function $f(x)$ is defined as:

$$E[X] = \mu = \int_{-\infty}^{\infty} x f(x) dx$$

4.4.2 Variance

The **variance** of X measures the dispersion around the mean:

$$Var(X) = \sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$$

Equivalently:

$$\text{Var}(X) = E[X^2] - (E[X])^2$$

4.5 Moments

Non-centered moments: The n -th non-centered moment of X is:

$$m_n = E[X^n] = \int_{-\infty}^{\infty} x^n f(x) dx$$

Centered moments: The n -th centered moment of X (around the mean μ) is:

$$\mu_n = E[(X - \mu)^n] = \int_{-\infty}^{\infty} (x - \mu)^n f(x) dx$$

4.6 Notes

- The first centered moment $\mu_1 = 0$.
- The second centered moment $\mu_2 = \text{Var}(X)$.
- Higher-order centered moments describe skewness (μ_3) and kurtosis (μ_4).

4.7 Covariance

For two continuous random variables X and Y with joint density $f_{X,Y}(x, y)$, the covariance is defined as:

$$\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])]$$

Equivalently:

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y]$$

Where:

$$E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{X,Y}(x, y) dx dy$$

4.8 4. Properties

- $\text{Cov}(X, X) = \text{Var}(X)$
- $\text{Cov}(X, Y) = \text{Cov}(Y, X)$
- If X and Y are independent, then $\text{Cov}(X, Y) = 0$