Chapter I : Numerical Series

1.1 Generalities

Let u_n be a sequence of real numbers. We set:

$$S_0 = u_0,$$

$$S_1 = u_0 + u_1,$$

$$S_2 = u_0 + u_1 + u_2,$$

$$\ldots\ldots..$$

$$S_n = u_0 + u_1 + u_{2+\cdots\ldots} + u_n = \sum_{k=0}^{n} u_k.$$

Definition 1

* The sequence $(S_n)_n$ is called the sequence of partial sums.

* The limit of (S_n) is called a series with general term .

Notation. A series with general term u_n is denoted: $\sum_{n=0}^{+\infty} u_n$ or $\sum_{n\geq 0} u_n$ or simply $\sum u_n$

1.1.1 Convergence

Definition 2 (Convergence).

If $(S_n)_n$ converges to S the series $\sum_{n=0}^{+\infty} u_n$ is said to be convergent, and

$$S = \lim_{n\to\infty} S_n = \sum_{n=0}^{+\infty} u_n$$

is its sum.

A series which is not convergent is said to be divergent.

The number $r_n = S - S_n = u_{n+1} + u_{n+2}u + \ldots\ldots$.is called the remainder of order n.

Example 1

(a) Geometric Series.

The general term of a geometric series is $u_n = a.r^n$, $a \neq 0$.

The partial sum is $S_n = \begin{cases} a.\frac{1-r^{n+1}}{1-r}, & r\neq 1 \\ a(n+1), & r=1 \end{cases}$

The series $\sum_{n=0}^{+\infty} u_n$is convergent if |r| < 1, and divergent if |r| > 1.

In the case of convergence: $\sum_{n=0}^{+\infty} u_n = \frac{a}{1-r}$

b) Harmonic Series

The general term of a harmonic series is $u_n = \frac{1}{n}, \quad n \in \mathbb{N}^*$

The series $\sum_{n=0}^{+\infty} \frac{1}{n}$ is a divergent series. We write $\sum_{n=0}^{+\infty} \frac{1}{n} = +\infty$

c) $\sum_{n=0}^{+\infty} u_n$, $\quad u_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$**.** Its partial sum is

$$S_n = u_1 + u_2 + u_3 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots .. + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

We have

$\lim_{n\to\infty} S_n$=1, thus the series $\sum_{n=0}^{+\infty} u_n$ is convergent and its sum is $S = 1$. We write $\sum_{n=1}^{+\infty} u_n = 1$

Necessary Condition for Convergence

If $\sum u_n$is convergent, then $\lim_{n\to\infty} u_n$=0

Remark

a) The condition $\lim_{n\to\infty} u_n$=0 is not sufficient.

b) If $\lim_{n\to\infty} u_n \neq 0$ then $\sum u_n$ is divergent.

Definition

If $\lim_{n\to\infty} u_n \neq 0$ the series $\sum u_n$ is said to be grossly divergent.

Example 2

a) We have $\lim_{n\to\infty} \frac{1}{n}$=0, but the series $\sum_{n=1}^{+\infty} \frac{1}{n}$ diverges (not grossly).

b) The series $\sum_{n=1}^{+\infty} cos\frac{\pi}{n}$ is grossly divergent because $\lim_{n\to\infty} cos\frac{\pi}{n}$ = 1$\neq$ 0.

1.1.2 Properties

If the series $\sum u_n$ differ only by a finite number of terms, then both series have the same nature. In the case of convergence, they do not necessarily have the same sum.

Proposition

If $\sum_{n=0}^{+\infty} u_n$= U and $\sum_{n=1}^{+\infty} v_n$= V are convergent, then $\sum_{n=0}^{+\infty}(\alpha u_n + \beta v_n)$ for all$\alpha, \beta \in R$, converges and

$$\sum_{n=0}^{+\infty}(\alpha u_n + \beta v_n) = \alpha U + \beta V$$

Example

Consider the series: $\sum_{n=0}^{+\infty}(\frac{3}{2^n} + \frac{2}{n(n+1)})$. This series is convergent because the series

$\sum_{n=0}^{+\infty} \frac{3}{2^n}$ and $\sum_{n=0}^{+\infty} \frac{2}{n(n+1)}$. both converge. Moreover, we have:

$$\sum_{n=0}^{+\infty}(\frac{3}{2^n} + \frac{2}{n(n+1)})=3\sum_{n=0}^{+\infty} \frac{3}{2^n} + 2\sum_{n=0}^{+\infty} \frac{2}{n(n+1)}=3\times 1 + 2 \times 1 = 5$$

Series with Positive Terms

Definition

A series $\sum u_n$ is called a series with positive terms if $u_n \geq 0$ for all $n \geq N_0$, where N_0, $\in \mathbb{N}$.

Example

The series: $\sum_{n=1}^{+\infty} \frac{n-5}{n^2}$ is a series with positive terms.

Proposition

A series with positive terms $\sum u_n$ converges if and only if the sequence of partial sums $(S_n)_n$is bounded above.

Comparison Rules

Theorem (Comparison).

Let $\sum u_n$ and $\sum v_n$ be two series with positive terms. Suppose that $0 \leq u_n \leq v_n$ for all $n \in \mathbb{N}$ .Then:

* If $\sum v_n$ converges,$\Rightarrow \sum u_n$ converges.
* If $\sum u_n$ diverges, then $\sum v_n$ diverges.

Example .

Consider the series : $\sum_{n=0}^{+\infty} \sin(\frac{1}{2^n})$ and $\sum_{n=0}^{+\infty} \frac{1}{2^n}$.

We have $0 \leq sin(\frac{1}{2^n}) \leq \frac{1}{2^n}$, and $\sum_{n=0}^{+\infty} \frac{1}{2^n}$ is a convergent geometric series, it follows that $\sum_{n=0}^{+\infty} \sin(\frac{1}{2^n})$ converges.

Theorem (equivalence criterion).

Let $\sum u_n$ and $\sum v_n$ be two series with strictly positive terms. Suppose that

$$\lim_{n\to\infty} \frac{u_n}{v_n} = l \text{ , } l \neq 0, l \neq +\infty$$

So, the two series $\sum u_n$ and $\sum v_n$ $are\ of\ the\ some\ nature.$

Example .

1. Let the series $\sum_{n=0}^{+\infty} u_n\ and\ \sum_{n=0}^{+\infty} v_n$ be such that $u_n = \log\left(1 + \frac{1}{2^n}\right) and\ v_n = \frac{2}{2^n}$

We have $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{3}$ and since $\sum_{n=0}^{+\infty} v_n$ is convergent, it follows that $\sum_{n=0}^{+\infty} u_n$ is also convergent.

2. Let the series $\sum_{n=0}^{+\infty} u_n\ and\ \sum_{n=0}^{+\infty} v_n$ be such that $u_n = \frac{1}{n} and\ v_n = \log\left(1 + \frac{1}{n}\right)$

We have $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = 1$. The series $\sum_{n=0}^{+\infty} u_n$ is the harmonic series, which is divergent. Therefore, $\sum_{n=0}^{+\infty} v_n$ is also divergent.

Theorem 15 (Comparison with an Integral)

Let $f : [1, +\infty[\to R^{+}$ be a continuous, decreasing, and positive function.

We set $u_n = f(n)$ for $n \in N^{*}$. Then

$$\sum u_n \text{ converges} \Leftrightarrow \int_1^{+\infty} f(x) \text{ exists}$$

Example

1. Consider the function $f : [1, +\infty[\to R^{+}$defined by $f(x) = \frac{1}{x}$

We have $\int_1^t \frac{1}{x} = logt \ and \ \lim_{t\to\infty} \int_1^t \frac{1}{x} = +\infty$. Therefore $\sum_{n=1}^{+\infty} \frac{1}{n}$diverges.

2. Consider the function $f : [1, +\infty[\to R^{+}$ defined by $f(x) = \frac{1}{x(x+1)}$

The function $f(x)$ is continuous, decreasing, and positive.

We have $\int_1^t f(x)= log\frac{t}{t+1} - log\frac{1}{2} \ and \ since \ \lim_{t\to\infty} \int_1^{+\infty} f(x)dx = \log(2) < +\infty$. Therefore $\sum_{n=1}^{+\infty} \frac{1}{n(n+1)}$ is convergent.

1.2.2 Riemann Series

Definition

We call Riemann series $\sum_{n=1}^{+\infty} u_n$ is any series whose general term is of the form

$$u_n = \frac{1}{n^{\alpha}}, \quad n \geq 1 \ and \ \alpha \in \boldsymbol{R}.$$

Proposition

The Riemann series $\sum_{n=1}^{+\infty} \frac{1}{n^{\alpha}}$converges if and only if α > 1.

Example

The series $\sum_{n=1}^{+\infty} \frac{1}{\sqrt{n}} \ and \ \sum_{n=1}^{+\infty} \sqrt{n}$ are divergent Riemann series.

D'Alembert criterion

Proposition

(D'Alembert criterion) Let $\sum u_n$ be a series with strictly positive terms., and set $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$.

* If l < 1, then $\sum u_n$ converges.
* If l > 1, then $\sum u_n$ diverges .
* If $l = 1$, we can't conclude anything.

Example

1. Let the series $\sum_{n=1}^{+\infty} u_n$with general term $u_n = (a + \frac{1}{n^p})^n$, where $a > 0$ and $p > 0$

We have $\lim_{n\to\infty} \sqrt[n]{u_n}$=$\lim_{n\to\infty} \sqrt[n]{(a + \frac{1}{n^p})^n}$ = $\lim_{n\to\infty} (a + \frac{1}{n^p})^n = a$,

Thus, the series converges for $a < 1$ and diverges for $a > 1$.

If $a = 1$, nothing can be concluded by using Cauchy’s criterion.

But we have

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (1 + \frac{1}{n^p})' = \lim_{n\to\infty} \left[\left(1 + \frac{1}{n^p}\right)^{n^p}\right]^{n^{1-p}} = \begin{cases} 1 & if\ p > 1 \\ e & if\ p = 1 \\ +\infty & if\ 0 < p < 1 \end{cases},$$

Since the general term does not tend to zero, the series is divergent.

Series with arbitrary terms

Definition

A series with arbitrary terms is a series $\sum u_n$ whose terms may be positive or negative depending on the values taken by$n$.

Example

The series $\sum_{n=0}^{+\infty} cosn\ and\ \ \sum_{n=0}^{+\infty} \cos(n\frac{\pi}{2})$ are series with arbitrary terms.

1.3.1 Abel's criterion

Proposition (Abel's criterion).

Let $\sum u_n$ be a series with arbitrary terms. Suppose that $u_n = a_n b_n$, with $\_b_n > 0$, such that:

* The sequence $(b_n)_n$ is decreasing and tends to 0. $(b_{n+1} < b_n)\ \forall n \in \mathbb{N}\ and\ \lim_{n\to\infty} b_n = 0$

* There exists M > 0 such that for all $p, q\ \in \mathbb{N} : p \geq q \Rightarrow \left|a_q + a_{q+1} + \cdots .. + a_p\right| \leq M$, then

Then, the series $\sum u_n$ is convergent.

Example

Show that the series $\sum_{n=0}^{+\infty} \frac{(-1)^n}{\sqrt{n+1}}$ is convergent.

Let $a_n = (-1)^n$ and b_n=$\frac{1}{\sqrt{n+1}}$. $\quad (u_n = \frac{(-1)^n}{\sqrt{n+1}} = a_n b_n)$

* The sequence $(b_n)_n$ is decreasing, and $b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$.

* Moreover, $\left|a_q + a_{q+1} + \cdots .. + a_p\right| = |(-1)^q + (-1)^{q+1} + \cdots .. + (-1)^p| \leq 1 = M$

Therefore, the series $\sum_{n=0}^{+\infty} \frac{(-1)^n}{\sqrt{n+1}}$ is convergent.

1.3.2 Alternating Series

Definition

A series is said to be alternating if

$$u_n . u_{n+1} \leq 0$$

The general term u_n of an alternating series can be written in the form

$$u_n = (-1)^n v_n \quad or \quad u_n = (-1)^{n+1} v_n$$

with $v_n \geq 0$

In general, an alternating series is often denoted as

$$\sum (-1)^n |u_n|$$

Leibniz's Criterion

Proposition (Leibniz's criterion).

Let $\sum u_n$ be an alternating series. Suppose that the sequence $(|u_n|)_n$ is decreasing and that

$$\lim_{n\to\infty} |u_n| = 0$$

Then the series $\sum u_n$ is convergent. Moreover, for every integer $n \in \mathbb{N}$, we have

$$\left|\sum_{n=0}^{+\infty} u_n\right| \leq |u_0| \;\; and \;\; \left|\sum_{k=n+1}^{+\infty} u_k\right| \leq |u_{n+1}|$$

Example

Consider the alternating harmonic series

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots ..$$

Here,

$$u_n = \frac{(-1)^{n+1}}{n}, \quad and \; (|u_n|)_n = (\frac{1}{n})_n$$

which is decreasing and tends to 0.

Therefore, the series

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n}$$

is convergent. Moreover,

$$\left|\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n}\right| \leq |u_0| \;\; and \;\; \left|\sum_{k=n+1}^{+\infty} \frac{(-1)^{k+1}}{k}\right| \leq |u_{n+1}| = \frac{1}{n+1}.$$

The series $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$ is conditionally convergent.