Chap02: Harmonic Regime

1. Introduction

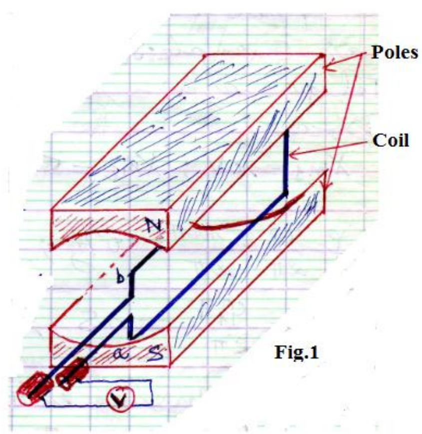
Alternating Current (AC) theory deals with the mathematical analysis of the steady-state behavior of linear electrical circuits in which currents and voltages are periodic functions of time. The simplest and most frequently encountered case—such as in domestic electrical power supplies—involves signals that vary sinusoidally, also known as harmonic variation.

2. Production of sinusoidal voltages

A coil of length **L** rotates at a constant angular speed ωt within a magnetic field generated by a pair of North and South poles. As the coil cuts through the magnetic field lines, a sinusoidal electromotive force $e(t)$ is induced, expressed as:

$$e(t) = E_{max} Sin\,(\omega t)$$

where: E_{max} represents the maximum (or peak) value of the induced voltage.



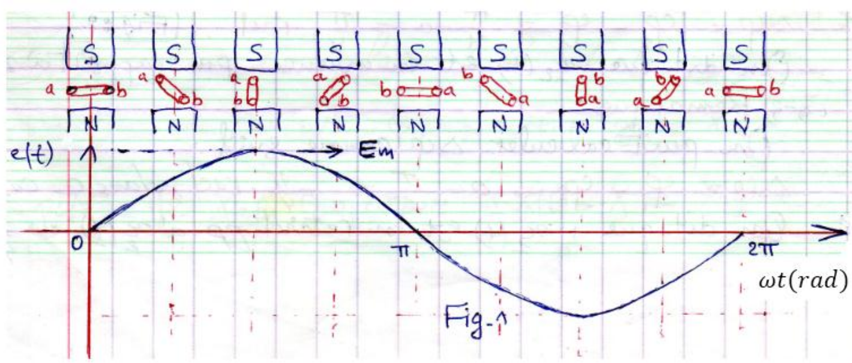


Figure1: Production of sinusoidal voltage

3. Temporal representation of sinusoidal quantities

3.1.Definition:

The instantaneous value of a sinusoidal signal can be written as:

$x(t) = X_m\, Cos(\omega t + \varphi)$

Or: $x(t) = X_m\, Sin(\omega t + \varphi)$

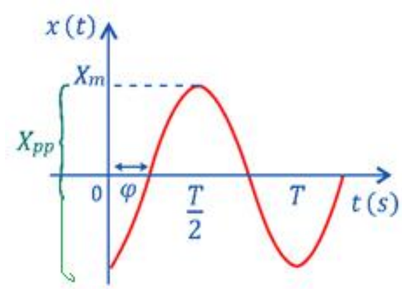


Figure 2: Graphical representation

Where:

X_m : Amplitude in (V) or (A)

ω: Pulsation in (rad/s)

$f = \dfrac{\omega}{2\pi} : Frequency\ in(Hz)$

$T = \frac{1}{f}\ Periode\ in(Hz)$

$X_{pp} = 2X_m :\ Peak\ to\ Peak\ value$

φ: $original\ phase\ in\ (rad)$

3.2.*Example*:

Let $x_1(t)$, $x_2(t)$ be two sinusoidal signals:

$x_1(t) = 10\ Cos(100\pi t)\ and\ x_2(t) = 5\ Cos(100\pi t + {}^{\pi}/_{4})$, So:

$X_{m_1} = 10V, X_{m_2} = 5V, X_{pp1} = 20V, X_{pp2} = 10V$ And $f = \frac{\omega}{2\pi} = \frac{100\pi}{2\pi} =$ 50 Hz , $\varphi_1 = 0\ rad, \varphi_2 = {}^{\pi}/_{4}\ rad$

The time difference: $\Delta t_1 = \frac{\varphi_1 T}{2\pi} = 0S, \Delta t_2 = \frac{\varphi_2 T}{2\pi} = 0{,}0025\ S$

The phase shift between $x_1(t)$ and $x_2(t)$ is:

$\Delta\varphi = \varphi_2 - \varphi_1 = \frac{\pi}{4} - 0 = 0\ rad$

In this case, we say that x_1 lags behind x_2 by π/4 rad.

Note: the phase shift can be calculated as follows:

$\Delta\varphi = \varphi_1 - \varphi_2 = 0 - \frac{\pi}{4} = -\frac{\pi}{4}\ rad$

In this case, we say that x_2 leads x_2 by π/4 rad.

We can use some helpful points to plot the two signals:

ωt°	0	90	180	270	360	450
$x_1(t)$	10	0	-10	0	10	0
ωt°	0	45	135	225	315	405
$x_2(t)$	3.54	0	-5	0	5	0

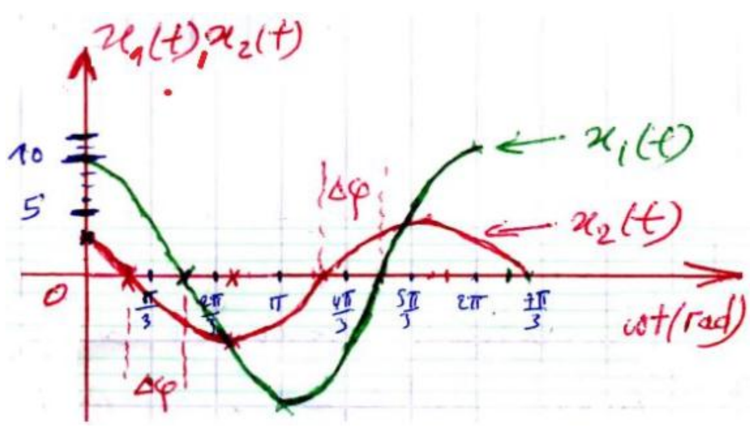


Figure 3: Graphical representation of $x_1(t)$ and $x_2(t)$

3.3. Average value of a periodic signal

$$X_{avg} = \int_0^T x(t)dt$$

Average value of a sinusoidal signal is: $X_{avg-sin} = 0$

3.4. RMS value of a periodic signal

$$X_{rms} = \sqrt{\frac{1}{T}\int_0^T x^2(t)dt} \quad , and\ X_{rms-sin} = \frac{X_m}{\sqrt{2}}$$

4. Vector representation or Fresnel representation

Let the two sinusoidal signals be expressed as:

$x(t) = \sqrt{2}\, X_{rms} Cos(\omega t)$

and $y(t) = \sqrt{2}\, Y_{rms} Cos(\omega t + \varphi)$

Where:

$\boldsymbol{X_{rms}}$ And $\boldsymbol{Y_{rms}}$: are the root mean square (RMS) values of the signals

$\boldsymbol{\omega}$: is the angular frequency in(rad/s)

$\boldsymbol{\varphi}$: is the phase difference between the two signals in (Radians).

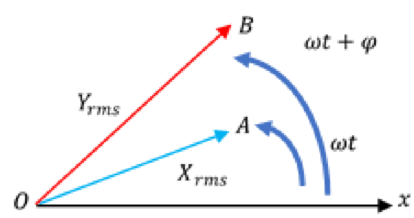


Figure 4: Vector representation of $x(t)$ and $y(t)$

- We assign the two vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ to the sinusoidal signals $x(t)$ and $y(t)$ respectively such that:

$\|\overrightarrow{OA}\| = X_{rms}$, $\|\overrightarrow{OB}\| = Y_{rms}$

And: $\quad (\overrightarrow{Ox}, \overrightarrow{OA}) = \omega t\ rad$, $(\overrightarrow{Ox}, \overrightarrow{OB}) = (\omega t + \varphi)\ rad$

Figure 4 shows the vector representation of the x and y signals.

The phase shift between the two vectors $\overrightarrow{OB}\ and\ \overrightarrow{OA}$ is denoted φ.

5. Simplified vector representation

To simplify the vector representation, we consider the image of the previous diagram at time:

$t = 0$

At this instant, the angular positions of the vectors are:

$(\overrightarrow{Ox}, \overrightarrow{OA}) = \ 0\,rad\,, (\overrightarrow{Ox}, \overrightarrow{OB}) = \varphi\,rad$

Thus:

- If $\varphi > 0$: the signal $y(t)$ is leads the signal $x(t)$
- If $\varphi < 0$: the signal $y(t)$ is lags behind the signal $x(t)$

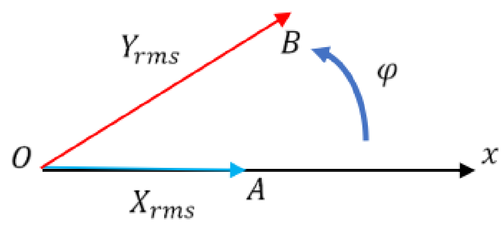


Figure 5: Simplified vector representation of $x(t)$ and $y(t)$

6. Complex Notation (phasor)

A sinusoidal signal can be expressed in complex exponential form as follows:

$$x(t) = X_m\,Cos(\omega t + \varphi) \Rightarrow \underline{x}(t) = \underline{x} = \ X_m e^{j(\omega t+\varphi)}$$

Or equivalently:

$$x(t) = \sqrt{2}\,X_{eff}\,Cos(\omega t + \varphi) \Rightarrow \underline{X}(t) = \underline{X} = \ X_{eff} e^{j(\omega t+\varphi)}$$

The quantities $X_m e^{j(\omega t+\varphi)}$ and $X_{eff} e^{j(\omega t+\varphi)}$ represent the complex form of the sinusoidal signal $x(t)$

7. Simplified Form

Since sinusoidal signals in the same circuit generally have the same angular frequency ω

We can simplify the complex representation by removing the time dependent-term ωt

$$\underline{x}(t) = \underline{x} = \ X_m e^{j\varphi}$$

Or:

$$\underline{X}(t) = \underline{X} = \ X_{eff} e^{j\varphi}$$

Finally, the real instantaneous value of the signal is obtained by taking the real part of the complex expression:

$$x(t) = Re\left\{\underline{x}(t)\right\}$$

7.1.Note:

For a sinusoidal quantity expressed in complex form:

$$\underline{x}(t) = \underline{x} = \underline{x}e^{j\omega t}$$

We can easily obtain its derivative and integral as follows:

$$\frac{d\underline{x}(t)}{dt} = \frac{d\underline{x}}{dt} = j\omega\ \underline{x}(t)$$

And:

$$\int \underline{x}(t)\ dt = \int \underline{x}\, dt = \frac{1}{j\omega}\ \underline{x}(t)$$

7.2.Example

Let us consider two sinusoidal signals:

$u_1(t) = \sqrt{2}\,.220\ Cos(\omega t + 30^{\circ})$

And:

$u_2(t) = \sqrt{2}\,.20\ Cos(\omega t - 70^{\circ})$

We are asked to calculate:

$u_1(t) + u_2(t)\ ;\ ;\ \frac{du_1(t)}{dt}\quad and\ \int u_2(t)$

7.3.Solution

1. We first express the signals in complex notation:

$\underline{u}_1 = \sqrt{2}\,.220\ e^{j30^{\circ}};\ \underline{u}_2 = \sqrt{2}\,.20\ e^{-j70^{\circ}}$

2. Sum of the two signals

$$\underline{u}_1(t) + \underline{u}_2(t) = \sqrt{2}\ (\ 220\ e^{j30^{\circ}} + 20\ e^{-j70^{\circ}}) = \sqrt{2}.\,217{,}4\ \ e^{j24.8^{\circ}}$$

3. Hence:

$$u_1(t) + u_2(t) = Re\left\{\sqrt{2}.\,217{,}4\ \ e^{j(\omega t+24.8^{\circ})}\right\} = \sqrt{2}\,.217{,}4\ Cos(\omega t + 24{,}8^{\circ})$$