TRANSIENT VIBRATIONS
OF
SDOF DYNAMIC STRUCTURES

In many cases, when a system is subject to a no-periodic excitation, the free vibration response interacts with the forced response and is important throughout the duration of the motion of the system. Such is the case when a system is subject to a pulse of finite duration where the period of free vibration is greater than the pulse duration.

One example of a non periodic excitation is the ground motion of an earthquake. The response of structures due to ground motion is obtained by using the methods of this chapter. An earthquake is usually of short duration, but maximum displacements and stresses occur while the earthquake takes place. The terrain traveled by a vehicle is usually non periodic. Suspension systems must be designed to protect passengers from sudden changes in road contour. Forces produced in operation of machines in manufacturing processes are often non periodic. Sudden changes in forces occur in presses and milling machines.

Forced vibrations of SDOF systems are described by the differential equation

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{1}{m_{eq}} F_{eq}(t) \tag{1}$$

The initial conditions and the homogeneous solution have an important effect on the short-term transient motion of vibrating systems. For these problems, it is convenient to use a solution method in which the homogeneous solution and particular solution are obtained simultaneously and the initial conditions are incorporated in the solution.

The primary method of solution presented in this chapter is use of the convolution integral. The convolution integral is derived using the principle of impulse and momentum and linear superposition. It can also be derived by application of the method of variation of parameters. The convolution integral provides the most general closed-form solution of the differential equation (1). The initial conditions are applied in the derivation of the integral, and need not be applied during every application. The convolution integral can be used to generate a unified mathematical response for excitations whose form changes at discrete times. Since it only requires evaluation of an integral, it is easy to apply.

A second method presented in this chapter is the convenient to use a solution method. Initial conditions are applied during the solution procedure.

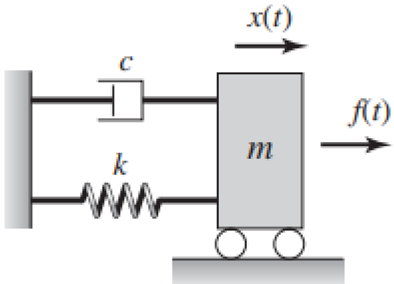

Figure .1. Spring-mass-damper system subjected to Transient Excitation *f*(t).

Solution for Response to Transient Excitation

When the initial displacement Xo = 0 and the initial velocity Vo = 0, the governing equation for an underdamped system ($0 < \zeta < 1$) with a forcing acting on the mass as shown in Figure 1 is given by Equation (2), which is repeated below.

$$x(t) = \frac{1}{m\omega_d}\int_0^t e^{-\zeta\omega_n\eta}\sin(\omega_d\eta)f(t-\eta)d\eta = \frac{1}{m\omega_d}\int_0^t e^{-\zeta\omega_n(t-\eta)}\sin(\omega_d[t-\eta])f(\eta)d\eta$$

η is the variable of integration and it is recalled that the damped natural frequency ω_d is given by:

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

In this chapter, we shall show how to:

• Analyze single degree-of-freedom systems subjected to abruptly changing forces, such as shocks, pulses, step functions, and ramps that are applied to the inertia element and to the base of the system.

• Study the significance of the ratio of the duration of a time-varying force to the period of free oscillation of the system.

• Relate the duration of a shock to its frequency content and study how this affects the system response

Physically, a pulse of zero duration does not exist and every pulse has a finite duration t1, very short. Its temporal variation during the duration t can be, for example sinusoidal, triangular or be represented by a "rectangular" slot (figure 2.) These situations are encountered for example in the case of a soft impact, an explosion or a hard impact.

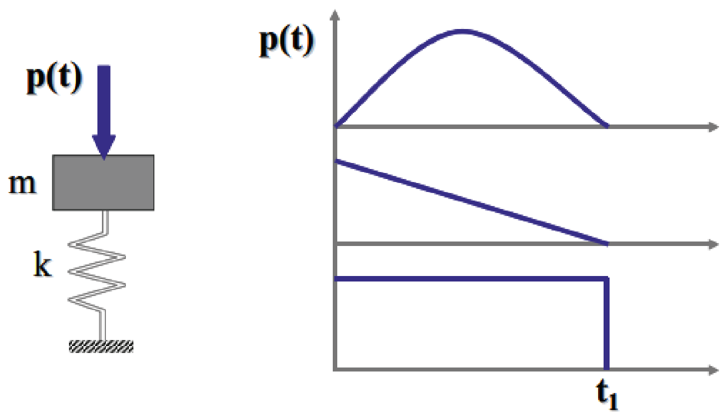

Figure .2. Impulse examples

The three principal types of transient forcing are :

1. Rectangular (step) pulse excitation called as the hard shock
2. Triangular or ramp pulse excitation seems like the explosion form
3. Sine pulse excitation called as the soft impact

A. Rectangular (step) pulse excitation:

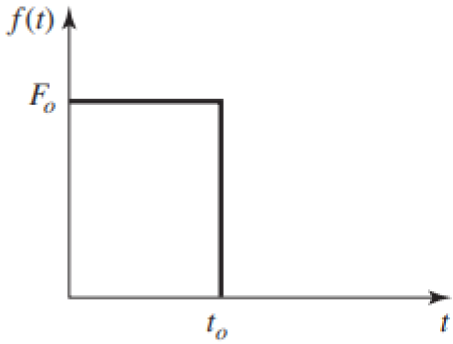

Figure .3. Rectangular force pulse.

Let we write the functions of the load pulse:

$$F(t) = \begin{cases} F_0 \; if \;\; t \le t_0 \\ 0 \;\; if \; t > t_0 \end{cases}$$

Case of undamped dynamic structure subject to rectangular pulse:

Consider a simple spring–mass system subjected to a rectangular step load as shown in Figure .3. We would like to find the response of the mass to this transient loading condition.

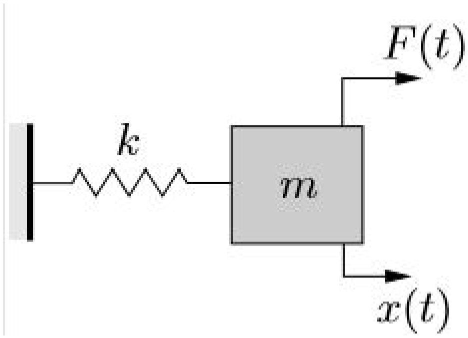

Figure.4. Mass-spring dynamic structure

The solution for undamped mass-spring structure by using **the convolution integral method** is:

For t < t_0

$$x(t) = \frac{1}{m\omega_n} \int_0^t F_0 \sin \omega_n (t - \tau) d\tau = \frac{F_0}{m\omega_n^2} \; (1 - \cos \omega_n t)$$

The same solution is obtained using **the initial conditions and particular solution method:**

The particular solution is : $x_p(t) = \frac{F_0}{k}$

The general response of the system is : $x_1(t) = \;\; \rho \cos(\omega_n t + \varphi) + \frac{F_0}{k}$

By imposing the initial conditions on the displacement (x(o)=0) and velocity ($\dot{x}(t) = 0$), the response will be:

$$x_1(t) = \frac{F_0}{k} \; (1 - \cos \omega_n t)$$

Which is valid for t< t_0.

At the moment when t=t_0 the oscillator has an initial displacement x(t_0) and an initial velocity $\dot{x}(t_0)$ to the next step where t> t_0.

$$x_2(t) = A\ \cos\omega_n(t-t_0) + \mathrm{B}\sin\omega_n(t-t_0)$$

$$x_2(t) = x_1(t_0).\cos\omega_n(t-t_0) + \frac{\dot{x_1}(t_0)}{\omega_n}\sin\omega_n(t-t_0)$$

$$x_2(t) = \frac{F_0}{k}\,(1-\cos\omega_n t_0).\cos\omega_n(t-t_0) + \frac{F_0}{k}.\sin\omega_n t_0.\sin\omega_n(t-t_0)$$

Valid for t> t_0

Search for the maximum displacement:

The maximum response during the free vibration phase is given by:

$$\rho_{max} = \sqrt{[x_1(t_0)]^2 + \left[\frac{x_1(\dot{t_0})}{\omega_n}\right]^2}$$

$$\rho_{max} = \frac{F_0}{K}.\sqrt{(\sin\omega_n t_0)^2 + (1-\cos\omega_n t_0)^2}$$

$$= \frac{F_0}{K}.\sqrt{(\sin\omega_n t_0)^2 + (\cos\omega_n t_0)^2 + 1 - 2.\cos\omega_n t_0} = \frac{F_0}{K}.\sqrt{2(1-\cos\omega_n t_0)}$$

$$= \frac{F_0}{K}.\sqrt{2.2.\sin^2(\frac{\omega_n t_0}{2})} = \frac{2.F_0}{K}.\sin(\frac{\omega_n t_0}{2})$$

We replace: $\omega_n = \frac{2\pi}{T}$ then we find $\rho_{max} = \frac{2F_0}{K}.\sin(\frac{\pi t_0}{T})$ which is in maximum value if only if : $\sin\left(\frac{\pi t_0}{T}\right) = 1$ ➔ $\frac{\pi t_0}{T} = \frac{\pi}{2}$ then $\sin\left(\frac{\pi t_0}{T}\right) < 1$ if $\frac{t_0}{T} < 0.5$

- For $\frac{t_0}{T} \leq 0.5$, the maximum response is obtained during the impulsive phase and is given by: $\rho_{max} = \frac{F_0}{k}\,(1-\cos(\frac{2\pi t_0}{T})\,) = \frac{2F_0}{K}.\sin(\frac{\pi t_0}{T})$
- For $\frac{t_0}{T} > 0.5$, the maximum response is obtained during the free vibration phase and is equal to 2.

The curve $D = \frac{k.\rho_{max}}{F_0}$ is represented as a function of t_0/T in Figure .9.

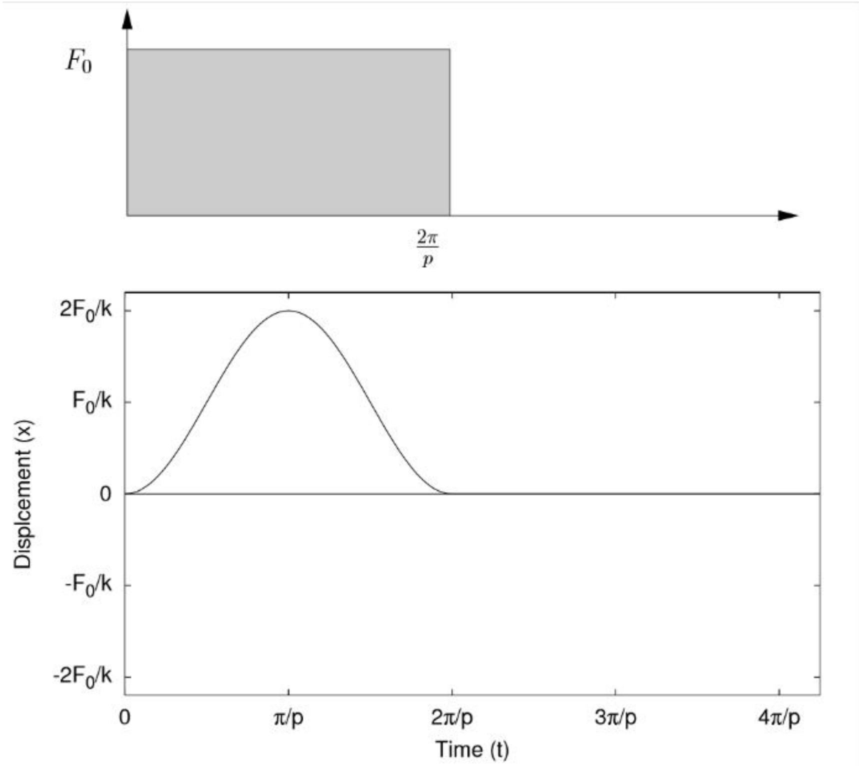

Figure.5. Response of simple spring–mass system to applied step load which is subsequently removed (a) $\tau = 2\pi/p$

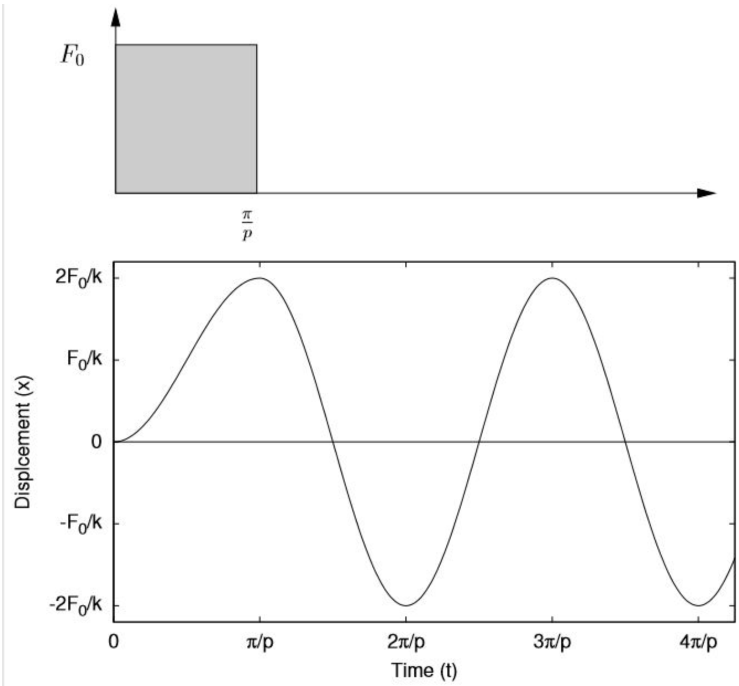

Figure .5. Response of simple spring–mass system to applied step load which is subsequently removed (b) $\tau = \pi/p$.

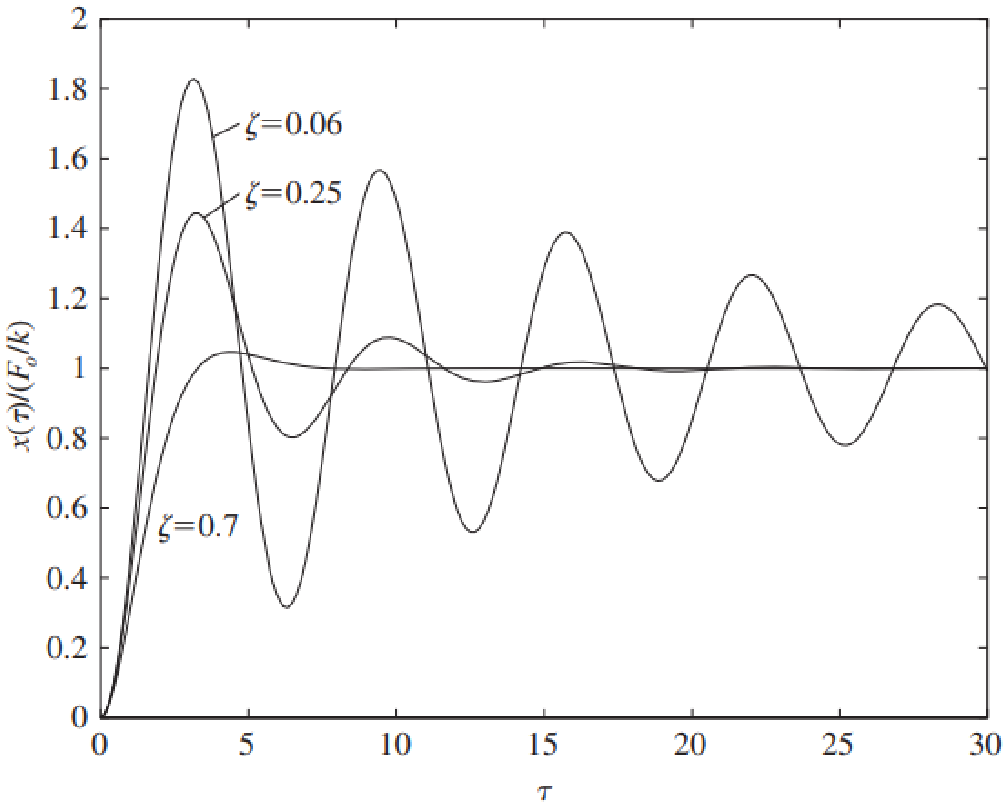

Figure .6. Responses of single degree-of-freedom systems with different damping factors to step inputs.

B. Triangular or ramp pulse excitation:

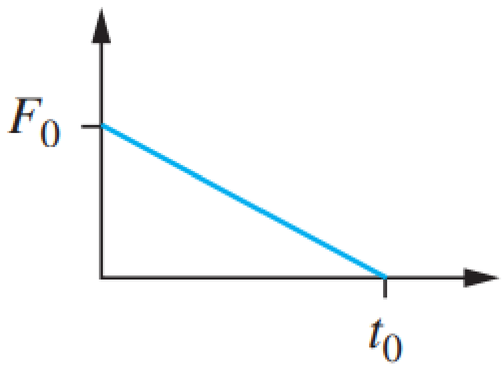

Figure .7. Ramp force composed of two ramps.

Let we write the functions of the load pulse:

$$F(\tau) = \begin{cases} F_0.(1-\frac{\tau}{t_0})\; if\;\; t \le t_0 \\ 0\;\; if\; t > t_0 \end{cases}$$

The response for $t \le t_0$ is given:

$$x_1(t) = \frac{1}{\omega_n t_0} sin\omega_n \frac{t}{t_0} - \cos\omega_n \frac{t}{t_0} + 1 - \frac{t}{t_0}$$

Taking the first derivative and setting it to zero, one can see that the first maximum occurs for $t = t_0$ for $t_0 = 0.37101T$, and substituting one can see that Rmax = 1.

C. Half-sine pulse excitation:

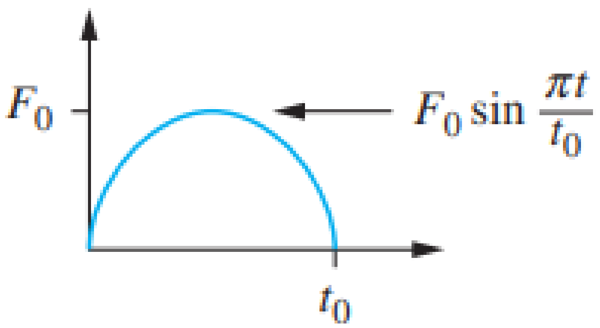

Figure .8. Half-sine force pulse.

Let we write the functions of the load pulse:

$F(\tau) = \begin{cases} F_0 . \sin \frac{\pi \tau}{t_0} \ \ if \ \ t \le t_0 \\ \quad 0 \ \ if \ t > t_0 \end{cases}$ in this case we have: $\overline{\omega} = \frac{\pi}{t_0}$

Consider an undamped SDOF initially at rest, with natural circular frequency ωn and stiffness k. With reference to a half-sine impulse with duration t0, the frequency ratio β is $\frac{\overline{\omega}}{\omega_n} = \frac{\pi T}{t_0 . 2\pi} = \frac{T}{2t_0}$

For $t \le t_0$ its response is:

$$x_1(t) = \frac{1}{1-\beta^2}\left(\sin \overline{\omega} t - \beta \sin \frac{\overline{\omega} t}{\beta}\right)$$

For $t > t_0$ its response is:

$$x_2(t) = \frac{-\beta}{1-\beta^2}\left(\left(1 + cos \frac{\pi}{\beta}\right)\sin \omega_n (t - t_0) + \sin \frac{\pi}{\beta} \cos \ \ \omega_n (t - t_0)\right)$$

Searching the maximum displacement:

Since we are interested in the maximum response ratio during the excitation, we need to know when velocity is zero.

$$\dot{x}_1(t) = \frac{\overline{\omega}}{1-\beta^2} \left(\cos \overline{\omega} t - \cos \frac{\overline{\omega} t}{\beta}\right) = 0$$

Substituting $\alpha = \frac{2n\beta}{\beta \mp 1}$;

In summary, to find the maximum of the response for an assigned β < 1, one has (a) to compute all $\alpha_k = \frac{2k\beta}{\beta+1}$ until a root is greater than 1, (b) compute all the responses for $t_k = \alpha_k . t_0$, (c) choose the maximum of the maxima.

For β > 1, the maximum response takes place for $t > t_0$, and its absolute value is:

$$R_{\max} = \frac{2\beta}{1-\beta^2} \cos \frac{\pi}{2\beta}.$$

$$x_{2,max} = \frac{2\beta}{1-\beta^2} \cos \frac{\pi}{2\beta}.$$

Conclusion:

For these different excitations, an explicit analytical solution can be obtained for the oscillator response. However, it is more fruitful to focus on the maximum response of the oscillator. This response can occur during the duration of the pulse or during the free vibration phase after the pulse ends.

In all cases, the maximum response is reached very quickly and the damping forces do not have the time necessary to absorb significant energy; it is therefore legitimate to be interested in the maximum response of the undamped oscillator (figure .2.). For a force of maximum amplitude p0, we will write that the maximum displacement is equal to:

$$\boxed{\max_{t} \mathbf{x}(t) = \frac{\mathbf{F}_0}{k} D}$$

Where D represents the maximum dynamic amplification coefficient. In general, whatever the shape of the pulse, D only depends on the ratio t_1/T of the duration of the pulse to the natural period of the oscillator. We can therefore represent the variation of D as a function of this parameter.

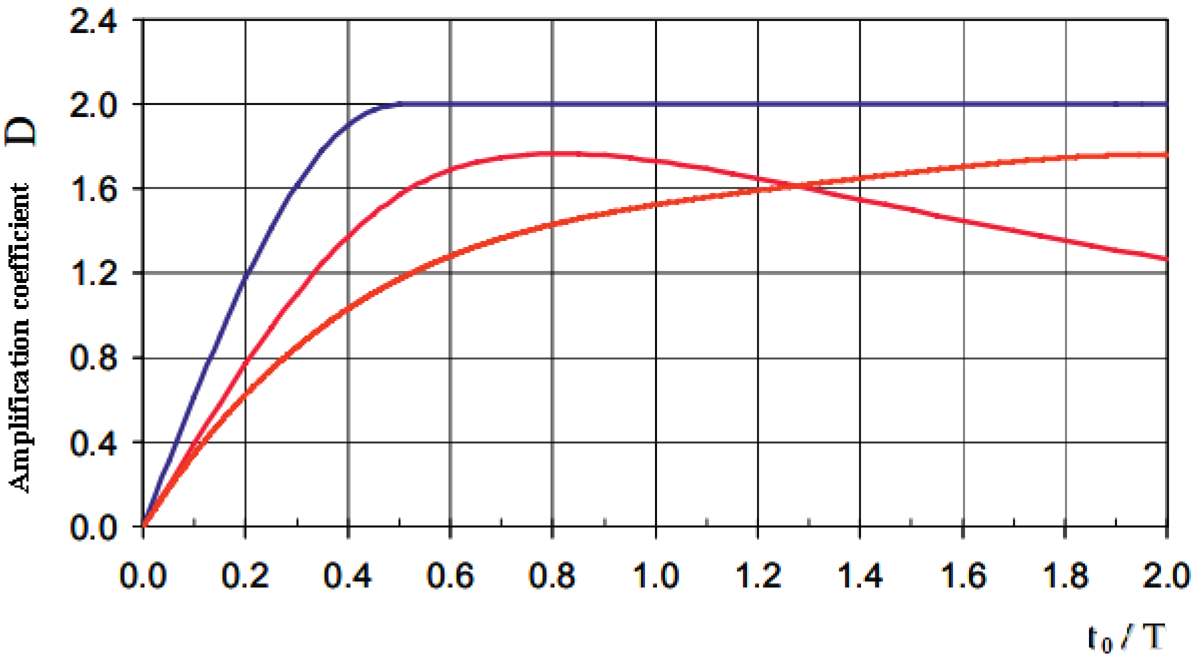

Figure .9. Shock spectra

Figure .9. shows this variation for the three pulse types in Figure .2. A graph of this type is called a shock spectrum.
We see in Figure .9. that the value of D is at most equal to 2.0, which justifies that in impact analyzes it is often recommended to use an equivalent static force equal to twice the applied force.

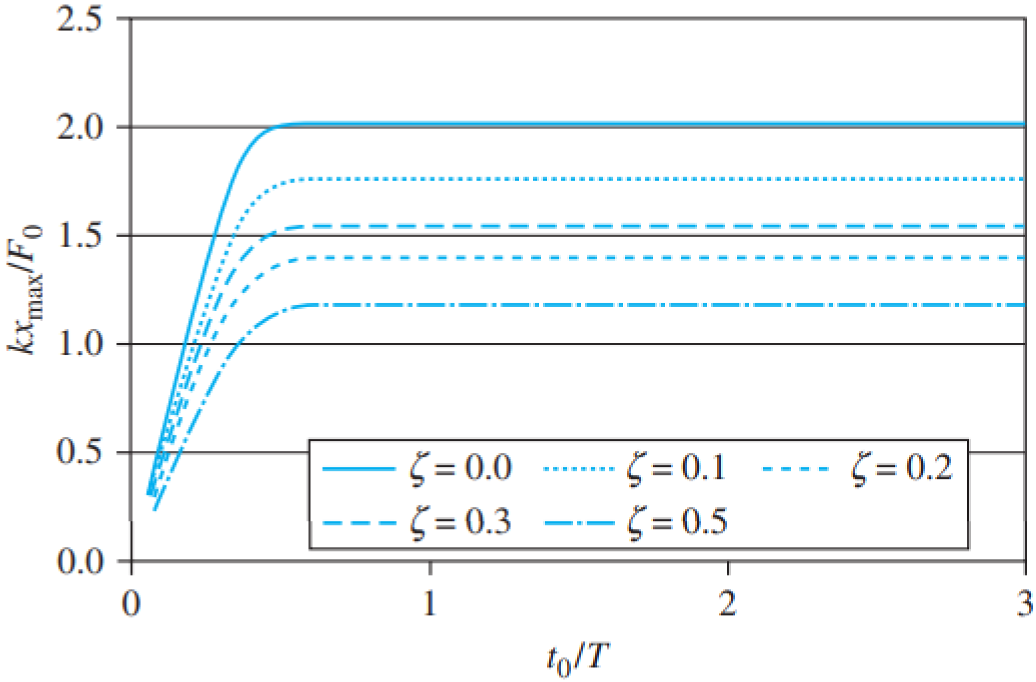

Figure.10. Response spectrum of 1D.O.F dynamic structure with different damping ratio for a rectangular pulse.

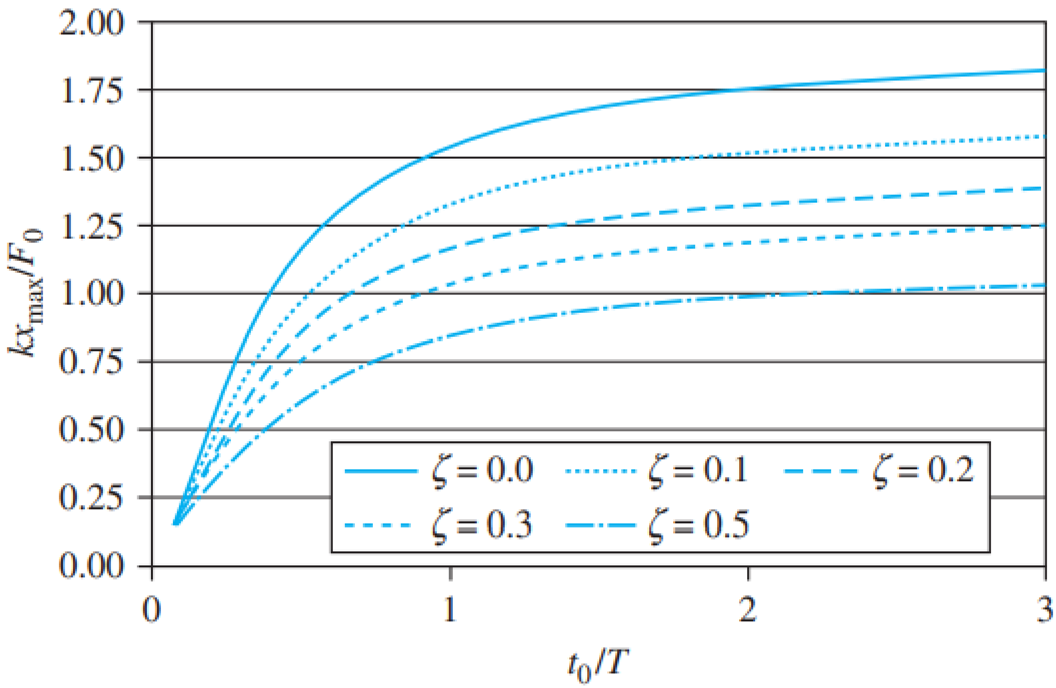

Figure.11. Response spectrum of 1D.O.F dynamic structure with different damping ratio for a triangular pulse.

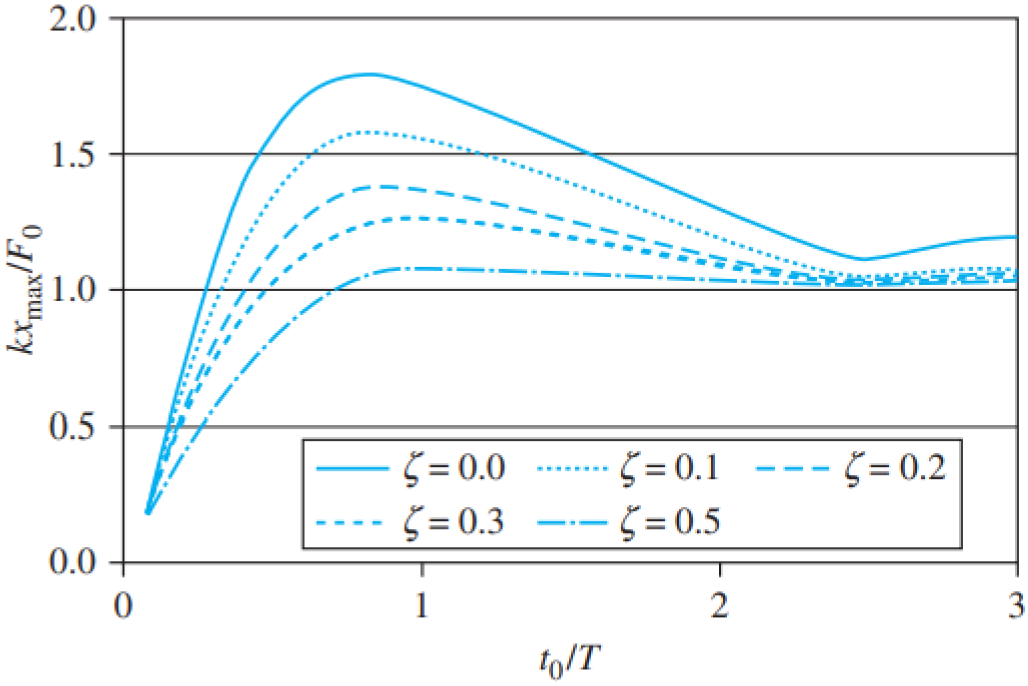

Figure.12. Response spectrum of 1D.O.F dynamic structure with different damping ratio for a half-sinusoidal pulse.