Chapter III. Shafts and Axles

III.1. Definitions

Since shafts take on several different configurations and are used for many different purposes, several different definitions are in common use, as listed below. The nomenclature is not always clear cut and there is often an overlap of function and therefore of definition.

- **Shaft** In general, a **ROTATING** member used for the transmission of power.
- **Axle** Generally a **STATIONARY** member used as a support for rotating members such as bearings, wheels, idler gears, etc.
- **Spindle** A short shaft, usually of small diameter, usually rotating, e.g. valve spindle for gate valve, but consider also the **HEADSTOCK SPINDLE** of a lathe, which is quite large and usually has a hole right through its centre
- **Stub shaft** A shaft which is integral with an engine, motor or prime mover and is of suitable size, shape and projection to allow its easy connection to other shafts.
- **Line shaft** (or power transmission shaft) A shaft connected to a prime mover which transmits power to a number of machines – now mostly superseded by machines having individual motors.
- **Jack shaft** A short shaft used to connect a prime mover to a machine or another shaft. May also be a short shaft placed as an intermediate shaft between a prime mover and driven machine.
- **Flexible shaft** Permits the transmission of power between two shafts (e.g. motor shaft and machine shaft) whose rotational axes are at an angle or where the angle between the shafts may change.

III.2. Shafts

A shaft is a rotating machine element which is used to transmit power from one place to another. The power is delivered to the shaft by some tangential force and the resultant torque (or twisting moment) set up within the shaft permits the power to be transferred to various machines linked up to the shaft. In order to transfer the power from one shaft to another, the various members such as pulleys, gears etc., are mounted on it. These members along with the forces exerted upon them causes the shaft to bending.

In other words, we may say that a shaft is used for the transmission of torque and bending moment. The various members are mounted on the shaft by means of keys or splines.

III.2. 1. Material Used for Shafts

The material used for shafts should have the following properties :
1. It should have high strength.
2. It should have good machinability.
3. It should have low notch sensitivity factor.
4. It should have good heat treatment properties.
5. It should have high wear resistant properties.
The material used for ordinary shafts is carbon steel of grades **40 C 8**, **45 C 8**, **50 C 4** and **50 C 12**.

III.2.2. Types of Shafts

The following two types of shafts are important from the subject point of view :
1. *Transmission shafts*. These shafts transmit power between the source and the machines absorbing power. The counter shafts, line shafts, over head shafts and all factory shafts are transmission shafts. Since these shafts carry machine parts such as pulleys, gears etc., therefore they are subjected to bending in addition to twisting.
2. *Machine shafts*. These shafts form an integral part of the machine itself. The crank shaft is
an example of machine shaft.

III.2.3. Stresses in Shafts

The following stresses are induced in the shafts :
1. Shear stresses due to the transmission of torque (*i.e.* due to torsional load).
2. Bending stresses (tensile or compressive) due to the forces acting upon machine elements like gears, pulleys etc. as well as due to the weight of the shaft itself.
3. Stresses due to combined torsional and bending loads.

III.2.4. Maximum Permissible Working Stresses for Transmission Shafts

According to American Society of Mechanical Engineers (ASME) code for the design of transmission shafts, the maximum permissible working stresses in tension or compression may be taken as
(*a*) 112 MPa for shafts without allowance for keyways.
(*b*) 84 MPa for shafts with allowance for keyways.

For shafts purchased under definite physical specifications, the permissible tensile stress (σ_t) may be taken as 60 per cent of the elastic limit in tension (σ_{el}), but not more than 36 per cent of the ultimate tensile strength (σ_u). In other words, the permissible tensile stress,
$\sigma_t = 0.6\ \sigma_{el}$ or $0.36\ \sigma_u$, whichever is less.

The maximum permissible shear stress may be taken as
(*a*) 56 MPa for shafts without allowance for key ways.

(*b*) 42 MPa for shafts with allowance for keyways.
For shafts purchased under definite physical specifications, the permissible shear stress (τ) may be taken as 30 per cent of the elastic limit in tension (σel) but not more than 18 per cent of the ultimate tensile strength (σu). In other words, the permissible shear stress, $\tau = 0.3\ \sigma el$ or $0.18\ \sigma u$, whichever is less.

III.2.5. Design of Shafts

The shafts may be designed on the basis of
1. Strength, and
2. Rigidity and stiffness.

In designing shafts on the basis of strength, the following cases may be considered :
(*a*) Shafts subjected to twisting moment or torque only,
(*b*) Shafts subjected to bending moment only,
(*c*) Shafts subjected to combined twisting and bending moments, and
(*d*) Shafts subjected to axial loads in addition to combined torsional and bending loads.

We shall now discuss the above cases, in detail, in the following pages.

III.2.6. Shafts subjected to twisting moment only

When the shaft is subjected to a twisting moment (or torque) only, then the diameter of the shaft may be obtained by using the torsion equation. We know that

$$\frac{T}{j} = \frac{\tau}{r}$$

where

- T = Twisting moment (or torque) acting upon the shaft,
- J = Polar moment of inertia of the shaft about the axis of rotation,
- τ = Torsional shear stress, and
- r = Distance from neutral axis to the outer most fibre
  = d / 2; where d is the diameter of the shaft.

We know that for round solid shaft, polar moment of inertia,

$J = (\pi. \times d^4)/ 32$

The equation may now be written as

$$\frac{T}{\frac{\pi}{32} x\, d^4} = \frac{\tau}{\frac{d}{2}} \ or \ \ T = \frac{\pi}{16} x\, \tau\, x d^3$$

From this equation, we may determine the diameter of round solid shaft (d). We also know that for hollow shaft, polar moment of inertia,

$$j = \frac{\pi}{32}[(d_0)^4 - (d_i)^4]$$

where do and di = Outside and inside diameter of the shaft, and $r = d_0 / 2$. Substituting these values in equation , we have

$$\frac{T}{\frac{\pi}{32}[(d_0)^4 - (d_i)^4]} = \frac{\tau}{\frac{d0}{2}} \ or \ T = \frac{\pi}{16} x\, \tau\, x \frac{[(d_0)^4 - (d_i)^4]}{d_0}$$

Let k = Ratio of inside diameter and outside diameter of the shaft = di / do
Now the equation may be written as

$$T = \frac{\pi}{16} . \tau . \frac{(d_0)^4}{d_0}\left[1 - \left(\frac{d_i}{d_0}\right)^4\right] = \frac{\pi}{16} . \tau\, (d_0)^3\, (1 - k^4)$$

From the last equations, the outside and inside diameter of a hollow shaft may be determined. It may be noted that

1. The hollow shafts are usually used in marine work. These shafts are stronger per kg of material and they may be forged on a mandrel, thus making the material more homogeneous than would be possible for a solid shaft.
When a hollow shaft is to be made equal in strength to a solid shaft, the twisting moment of both the shafts must be same. In other words, for the same material of both the shafts,

$$T = \frac{\pi}{16} . \tau \left[\frac{(d_0)^4 - (d_i)^4}{d_0}\right] = \frac{\pi}{16} . \tau . d^3$$

$$\frac{(d_0)^4 - (d_i)^4}{d_0} = d^3 \ or \ (d_0)^3 (1 - k^4) = d^3$$

2. The twisting moment (T) may be obtained by using the following relation : We know that the power transmitted (in watts) by the shaft,

$$P = \frac{2\pi N \,.\, T}{60} \quad or\; T = \frac{P\,.\,60}{2\pi\, N}$$

where

- T = Twisting moment in N-m, and
- N = Speed of the shaft in r.p.m.

3. In case of belt drives, the twisting moment (T) is given by
$T = (T1 – T2)\, R$

where

- $T1$ and $T2$ = Tensions in the tight side and slack side of the belt respectively, and
- R = Radius of the pulley.

III.2.7. Shafts subjected to Bending moment only

When the shaft is subjected to a bending moment only, then the maximum stress (tensile or compressive) is given by the bending equation. We know that

$$\frac{M}{y} = \frac{\sigma_b}{y}$$

where

- M = Bending moment,
- I = Moment of inertia of cross-sectional area of the shaft about the axis of rotation,
- σ_b = Bending stress, and
- y = Distance from neutral axis to the outer-most fibre.

We know that for a round solid shaft, moment of inertia,

$$I = \frac{\pi d^4}{64} \qquad and \qquad y = \frac{d}{2}$$

Substituting these values in last equation, we have

$$\frac{M}{\frac{\pi}{64}\,.\,d^4} = \frac{\sigma_b}{\frac{d}{2}} \quad or\;\; M = \frac{\pi}{32}\,.\,\sigma_b\,.\,d^3$$

From this equation, diameter of the solid shaft (d) may be obtained.

We also know that for a hollow shaft, moment of inertia,

$$I = \frac{\pi}{64}\left[(d_o)^4 - (d_i)^4\right] = \frac{\pi}{64}\,(d_o)^4\,(1-k^4)$$

where

- $k = di / do$)
- $y = do/ 2$

Again substituting these values in equation, we have

$$\frac{M}{\frac{\pi}{64}\,(d_o)^4\,(1-k^4)} = \frac{\sigma_b}{\frac{d_o}{2}} \qquad \text{or} \qquad M = \frac{\pi}{32}\times\sigma_b\,(d_o)^3\,(1-k^4)$$

From this equation, the outside diameter of the shaft (do) may be obtained.

III.2.7. Shafts subjected to combined twisting moment and Bending moment

When the shaft is subjected to combined twisting moment and bending moment, then the shaft must be designed on the basis of the two moments simultaneously. Various theories have been suggested to account for the elastic failure of the materials when they are subjected to various types of combined stresses. The following two theories are important from the subject point of view :

1. Maximum shear stress theory or Guest's theory. It is used for ductile materials such as mildsteel.
2. Maximum normal stress theory or Rankine's theory. It is used for brittle materials such as cast iron.
Let τ = Shear stress induced due to twisting moment, and σb = Bending stress (tensile or compressive) induced due to bending moment.

According to maximum shear stress theory, the maximum shear stress in the shaft,

$$\tau_{max} = \frac{1}{2}\sqrt{(\sigma_b)^2 + 4\tau^2}$$

The maximum B.M. may be obtained as follows :

- $RC = RD = 50$ kN $= 50 \times 10^3$ N
- B.M. at *A*, *M*A = 0
- B.M. at *C*, *M*C = $50 \times 10^3 \times 100 = 5 \times 10^6$ N-mm
- B.M. at *D*, *M*D = $50 \times 10^3 \times 1500 - 50 \times 10^3 \times 1400 = 5 \times 10^6$ N-mm
- B.M. at *B*, *M*B = 0

Substituting the values of τ and σ_b

$$\tau_{max} = \frac{1}{2}\sqrt{\left(\frac{32M}{\pi d^3}\right)^2 + 4\left(\frac{16T}{\pi d^3}\right)^2} = \frac{16}{\pi d^3}\left[\sqrt{M^2 + T^2}\right]$$

Or

$$\frac{\pi}{16} \times \tau_{max} \times d^3 = \sqrt{M^2 + T^2}$$

The expression $(M^2 + T^2)^{1/2}$ is known as ***equivalent twisting moment*** and is denoted by *Te*. The equivalent twisting moment may be defined as that twisting moment, which when acting alone, produces the same shear stress (τ) as the actual twisting moment. By limiting the maximum shear stress (τmax) equal to the allowable shear stress (τ) for the material, the last equation may be written as

$$T_e = \sqrt{M^2 + T^2} = \frac{\pi}{16} \times \tau \times d^3$$

From this expression, diameter of the shaft (*d*) may be evaluated.
Now according to maximum normal stress theory, the maximum normal stress in the shaft,

$$\begin{aligned}\sigma_{b(max)} &= \frac{1}{2}\sigma_b + \frac{1}{2}\sqrt{(\sigma_b)^2 + 4\tau^2}\\ &= \frac{1}{2} \times \frac{32M}{\pi d^3} + \frac{1}{2}\sqrt{\left(\frac{32M}{\pi d^3}\right)^2 + 4\left(\frac{16T}{\pi d^3}\right)^2}\\ &= \frac{32}{\pi d^3}\left[\frac{1}{2}(M + \sqrt{M^2 + T^2})\right]\end{aligned}$$

The expression $1/2\ [M + (M^2+T^2)^{1/2}]$ is known as ***equivalent bending moment*** and is denoted by *Me*. The equivalent bending moment may be defined as **that moment which when acting alone produces the same tensile or compressive stress (σ*b*) as the actual bending moment.** By limiting the maximum normal stress [$\sigma b(max)$] equal to the allowable bending stress (σb), then the equation may be written as

$$M_e = \frac{1}{2}\left[M + \sqrt{M^2 + T^2}\right] = \frac{\pi}{32} \times \sigma_b \times d^3$$

From this expression, diameter of the shaft (*d*) may be evaluated.

Notes:

1. In case of a hollow shaft, the last equations and may be written as

$$T_e = \sqrt{M^2 + T^2} = \frac{\pi}{16} \times \tau\, (d_o)^3\,(1 - k^4)$$

and

$$M_e = \tfrac{1}{2}\left(M + \sqrt{M^2 + T^2}\right) = \frac{\pi}{32} \times \sigma_b\, (d_o)^3\,(1 - k^4)$$

2. It is suggested that diameter of the shaft may be obtained by using both the theories and the larger of the two values is adopted.

III.2.8. Shafts subjected to Fluctuating loads

In the previous articles we have assumed that the shaft is subjected to constant torque and bending moment. But in actual practice, the shafts are subjected to fluctuating torque and bending moments. In order to design such shafts like line shafts and counter shafts, the combined shock and fatigue factors must be taken into account for the computed twisting moment (*T*) and bending moment (*M*). Thus for a shaft subjected to combined bending and torsion, the equivalent twisting moment,

$$T_e = \sqrt{(K_m \times M)^2 + (K_t + T)^2}$$

and equivalent bending moment,

$$M_e = \frac{1}{2}\left[K_m \times M + \sqrt{(K_m \times M)^2 + (K_t \times T)^2}\right]$$

Where

- *Km* = Combined shock and fatigue factor for bending, and
- *Kt* = Combined shock and fatigue factor for torsion.

The following table III.1. Shows the recommended values for *Km* and *Kt*.

Table III.1. Recommended values for *Km* and *Kt*.

Nature of load	K_m	K_t
1. Stationary shafts		
(*a*) Gradually applied load	1.0	1.0
(*b*) Suddenly applied load	1.5 to 2.0	1.5 to 2.0
2. Rotating shafts		
(*a*) Gradually applied or steady load	1.5	1.0
(*b*) Suddenly applied load with minor shocks only	1.5 to 2.0	1.5 to 2.0
(*c*) Suddenly applied load with heavy shocks	2.0 to 3.0	1.5 to 3.0

III.2.9. Shafts subjected to axial load in addition to combined torsion and bending loads

When the shaft is subjected to an axial load (*F*) in addition to torsion and bending loads as in propeller shafts of ships and shafts for driving worm gears, then the stress due to axial load must be added to the bending stress (*σb*). We know that bending equation is

$$\frac{M}{I} = \frac{\sigma_b}{y} \quad \text{or} \quad \sigma_b = \frac{M \cdot y}{I} = \frac{M \times d/2}{\frac{\pi}{64} \times d^4} = \frac{32M}{\pi d^3}$$

and stress due to axial load

$$= \frac{F}{\frac{\pi}{4} \times d^2} = \frac{4F}{\pi d^2}$$ for round solid shaft

$$= \frac{F}{\frac{\pi}{4}\left[(d_o)^2 - (d_i)^2\right]} = \frac{4F}{\pi\left[(d_o)^2 - (d_i)^2\right]}$$ For hollow shaft)

$$= \frac{F}{\pi (d_o)^2 (1 - k^2)}$$ $k = di/do$

Resultant stress (tensile or compressive) for solid shaft,

$$\sigma_1 = \frac{32M}{\pi d^3} + \frac{4F}{\pi d^2} = \frac{32}{\pi d^3}\left(M + \frac{F \times d}{8}\right)$$

$$= \frac{32M_1}{\pi d^3} \quad \left(\text{Substituting } M_1 = M + \frac{F \times d}{8}\right)$$

In case of a hollow shaft, the resultant stress,

$$\sigma_1 = \frac{32M}{\pi (d_o)^3 (1 - k^4)} + \frac{4F}{\pi (d_o)^2 (1 - k^2)}$$

$$= \frac{32}{\pi (d_o)^3 (1 - k^4)}\left[M + \frac{F d_o (1 + k^2)}{8}\right] = \frac{32M_1}{\pi (d_o)^3 (1 - k^4)}$$

$$\ldots\left[\text{Substituting for hollow shaft, } M_1 = M + \frac{F d_o (1 + k^2)}{8}\right]$$

In case of long shafts (slender shafts) subjected to compressive loads, a factor known as ***column factor*** (α) must be introduced to take the column effect into account.

$$\sigma_c = \frac{\alpha \times 4F}{\pi d^2}$$

Stress due to the compressive load,

$$= \frac{\alpha \times 4F}{\pi (d_o)^2 \, (1-k^2)}$$

The value of column factor (α) for compressive loads* may be obtained from the following relation:

Column factor,

$$\alpha = \frac{1}{1 - 0.0044 \, (L/K)}$$

This expression is used when the slenderness ratio (L / K) is less than 115. When the slenderness ratio (L / K) is more than 115, then the value of column factor may be obtained from the following relation :

Column factor,

$$\alpha = \frac{\sigma_y (L/K)^2}{C \pi^2 E}$$

where

- L = Length of shaft between the bearings,
- K = Least radius of gyration,
- σy = Compressive yield point stress of shaft material, and
- C = Coefficient in Euler's formula depending upon the end conditions.

Note: In general, for a hollow shaft subjected to fluctuating torsional and bending load, along with an axial load, the equations for equivalent twisting moment (Te) and equivalent bending moment (Me) may be written as

$$T_e = \sqrt{\left[K_m \times M + \frac{\alpha F \, d_o \, (1+k^2)}{8}\right]^2 + (K_t \times T)^2}$$
$$= \frac{\pi}{16} \times \tau \, (d_o)^3 \, (1-k^4)$$

And

$$M_e = \frac{1}{2}\left[K_m \times M + \frac{\alpha F d_o \, (1+k^2)}{8} + \sqrt{\left\{K_m \times M + \frac{\alpha F d_o \, (1+k^2)}{8}\right\}^2 + (K_t \times T)^2}\right]$$
$$= \frac{\pi}{32} \times \sigma_b \, (d_o)^3 \, (1-k^4)$$

It may be noted that for a solid shaft, k = 0 and $d0$ = d. When the shaft carries no axial load, then F = 0 and when the shaft carries axial tensile load, then α = 1.