Chapter 4- Material Behavior—Linear Elastic Solids

4.1. Material Characterization

Relations that characterize the physical properties of materials are called *constitutive equations*. Because of the endless variety of materials and loadings, the study and development of constitutive equations is perhaps one of the most interesting and challenging fields in mechanics. Although continuum mechanics theory has established some principles for systematic development of constitutive equations, many constitutive laws have been developed through empirical relations based on experimental evidence. Our interest here is limited to a special class of solid materials with loadings resulting from mechanical or thermal effects. The mechanical behavior of solids is normally defined by constitutive stress-strain relations. Commonly, these relations express the stress as a function of the strain, strain rate, strain history, temperature, and material properties. We choose a rather simple material model called the *elastic solid* that does not include rate or history effects. The model may be described as a deformable continuum that recovers its original configuration when the loadings causing the deformation are removed. Furthermore, we restrict the constitutive stress-strain law to be linear, thus leading to a *linear elastic solid*. Although these assumptions greatly simplify the model, linear elasticity predictions have shown good agreement with experimental data and have provided useful methods to conduct stress analysis. Many structural materials including metals, plastics, ceramics, wood, rock, concrete, and so forth exhibit linear elastic behavior under small deformations.

As mentioned, experimental testing is commonly employed in order to characterize the mechanical behavior of real materials. One such technique is the simple tension test in which a specially prepared cylindrical or flat stock sample is loaded axially in a testing machine. Strain is determined by the change in length between prescribed reference marks on the sample and is usually measured by a clip gage. Load data collected from a load cell is divided by the cros-ssectional area in the test section to calculate the stress. Axial stress-strain data is recorded and plotted using standard experimental techniques. Typical qualitative data for three types of structural metals (mild steel, aluminum, cast iron) are shown in Figure 4-1. It is observed that each material exhibits an initial stress-strain response for small deformation that is approximately linear. This is followed by a change to nonlinear behavior that can lead to large deformation, finally ending with sample failure.

For each material the initial linear response ends at a point normally referred to as the proportional limit. Another observation in this initial region is that if the loading is removed, the sample returns to its original shape and the strain disappears. This characteristic is the primary descriptor of elastic behavior. However, at some point on the stress-strain curve unloading does not bring the sample back to zero strain and some permanent plastic deformation results. The point at which this non-elastic behavior begins is called the *elastic limit*. Although some materials exhibit different elastic and proportional limits, many times these values are taken to be approximately the same. Another demarcation on the stress-strain curve is referred to as the *yield point*, defined by the location where large plastic deformation begins.

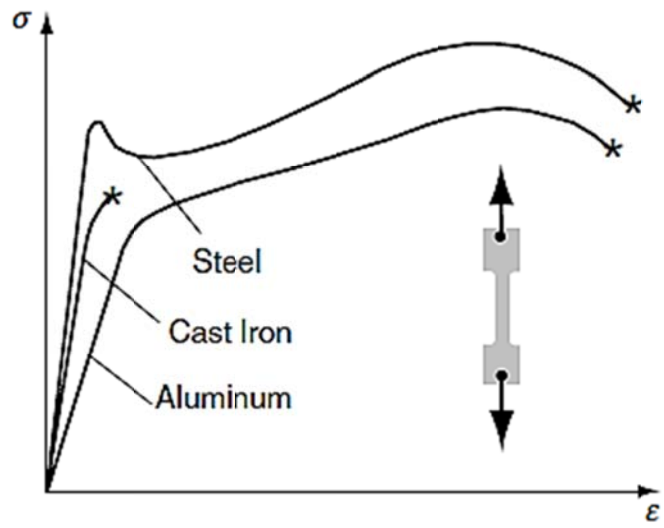

Figure 4-1 Typical uniaxial stress-strain curves for three structural metals.

Because mild steel and aluminum are ductile materials, their stress-strain response indicates extensive plastic deformation, and during this period the sample dimensions will be changing. In particular the sample's cross-sectional area undergoes significant reduction, and the stress calculation using division by the original area will now be in error. This accounts for the reduction in the stress at large strain. If we were to calculate the load divided by the true area, the *true stress* would continue to increase until failure. On the other hand, cast iron is known to be a brittle material, and thus its stress-strain response does not show large plastic deformation. For this material, very little non-elastic or nonlinear behavior is observed. It is therefore concluded from this and many other studies that a large variety of real materials exhibits linear elastic behavior under small deformations. This would lead to a linear constitutive model for the one-dimensional axial loading case given by the relation $\sigma = E\varepsilon$, where E is the slope of the uniaxial stress-strain curve. We now use this simple concept to develop the general three-dimensional forms of the linear elastic constitutive model.

4.2. Linear Elastic Materials—Hooke's Law

Based on observations from the previous section, in order to construct a general three-dimensional constitutive law for linear elastic materials, we assume that each stress component is linearly related to each strain component

$$\begin{aligned}
\sigma_x &= C_{11}e_x + C_{12}e_y + C_{13}e_z + 2C_{14}e_{xy} + 2C_{15}e_{yz} + 2C_{16}e_{zx} \\
\sigma_y &= C_{21}e_x + C_{22}e_y + C_{23}e_z + 2C_{24}e_{xy} + 2C_{25}e_{yz} + 2C_{26}e_{zx} \\
\sigma_z &= C_{31}e_x + C_{32}e_y + C_{33}e_z + 2C_{34}e_{xy} + 2C_{35}e_{yz} + 2C_{36}e_{zx} \\
\tau_{xy} &= C_{41}e_x + C_{42}e_y + C_{43}e_z + 2C_{44}e_{xy} + 2C_{45}e_{yz} + 2C_{46}e_{zx} \\
\tau_{yz} &= C_{51}e_x + C_{52}e_y + C_{53}e_z + 2C_{54}e_{xy} + 2C_{55}e_{yz} + 2C_{56}e_{zx} \\
\tau_{zx} &= C_{61}e_x + C_{62}e_y + C_{63}e_z + 2C_{64}e_{xy} + 2C_{65}e_{yz} + 2C_{66}e_{zx}
\end{aligned} \tag{4.1}$$

where the coefficients C_{ij} are material parameters and the factors of 2 arise because of the symmetry of the strain. Note that this relation could also be expressed by writing the strains as a linear function of the stress components. These relations can be cast into a matrix format as

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & \cdot & \cdot & \cdot & C_{16} \\ C_{21} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ C_{61} & \cdot & \cdot & \cdot & \cdot & C_{66} \end{bmatrix} \begin{bmatrix} e_x \\ e_y \\ e_z \\ 2e_{xy} \\ 2e_{yz} \\ 2e_{zx} \end{bmatrix} \quad \textbf{(4.2)}$$

Relations **(4.1)** can also be expressed in standard tensor notation by writing

$$\sigma_{ij} = C_{ijkl}\, e_{kl} \quad \textbf{(4.3)}$$

Where C_{ijkl} is a fourth-order elasticity tensor whose components include all the material parameters necessary to characterize the material. Based on the symmetry of the stress and strain tensors, the elasticity tensor must have the following properties

$$\begin{aligned} C_{ijkl} &= C_{jikl} \\ C_{ijkl} &= C_{ijlk} \end{aligned} \quad \textbf{(4.4)}$$

In general, the fourth-order tensor C_{ijkl} has 81 components. However, relations (**4.4**) reduce the number of independent components to 36, and this provides the required match with form **(4.1)** or **(4.2).** Later we introduce the concept of strain energy, and this leads to a further reduction to 21 independent elastic components. The components of C_{ijkl} or equivalently C_{ij} are called elastic moduli and have units of stress (force/area). In order to continue further, we must address the issues of material homogeneity and isotropy.

If the material is homogenous, the elastic behavior does not vary spatially, and thus all elastic moduli are constant. For this case, the elasticity formulation is straightforward, leading to the development of many analytical solutions to problems of engineering interest. A homogenous assumption is an appropriate model for most structural applications, and thus we primarily choose this particular case for subsequent formulation and problem solution. However, there are a couple of important nonhomogeneous applications that warrant further discussion.

Studies in geomechanics have found that the material behavior of soil and rock commonly depends on distance below the earth's surface. In order to simulate particular geomechanics problems, researchers have used nonhomogeneous elastic models applied to semi-infinite domains. Typical applications have involved modeling the response of a semi-infinite soil mass under surface or subsurface loadings with variation in elastic moduli with depth. Another more recent application involves the behavior of functionally graded materials (FGM). FGMs are a new class of engineered materials developed with spatially varying properties to suit particular applications. The graded composition of such materials is commonly established and controlled using powder metallurgy, chemical vapor deposition, or centrifugal casting. Typical analytical studies of these materials have assumed linear, exponential, and power-law variation in elastic moduli of the form

$$\begin{aligned} C_{ij}(x) &= C^o_{ij}(1+ax) \\ C_{ij}(x) &= C^o_{ij}e^{ax} \\ C_{ij}(x) &= C^o_{ij}x^a \end{aligned} \quad \textbf{(4.5)}$$

where C^o_{ij} and a are prescribed constants and *x* is the spatial coordinate.

Similar to homogeneity, another fundamental material property is isotropy. This property has to do with differences in material moduli with respect to orientation. For example, many materials including crystalline minerals, wood, and fiber-reinforced composites have different elastic moduli in different directions. Materials such as these are said to be anisotropic. Note that for most real anisotropic materials there exist particular directions where the properties are the same. These directions indicate *material symmetries*. However, for many engineering materials (most structural metals and many plastics), the orientation of crystalline and grain microstructure is distributed randomly so that macroscopic elastic properties are found to be essentially the same in all directions. Such materials with complete symmetry are called *isotropic.*

The tensorial form **(4.3)** provides a convenient way to establish the desired isotropic stress-strain relations. If we assume isotropic behavior, the elasticity tensor must be the same under all rotations of the coordinate system. Using the basic transformation properties from relation **(1.1),** the fourth-order elasticity tensor must satisfy

$$C_{ijkl} = Q_{im}Q_{jn}Q_{kp}Q_{lq}C_{mnpq}$$

It can be shown that the most general form that satisfies this isotropy condition is given by

$$C_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk} \quad \textbf{(4.6)}$$

where α, β, and γ are arbitrary constants**.** Using the general form (4.2.6) in stress-strain relation (4.3) gives

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} \quad \textbf{(4.7)}$$

where we have relabeled particular constants using λ and μ. The elastic constant l is called *Lame´´s* constant, and μ is referred to as the *shear modulus* or *modulus of rigidity.* Some texts use the notation *G* for the shear modulus. Equation (4.2.7) can be written out in individual scalar equations as

$$\begin{aligned} \sigma_x &= \lambda(e_x+e_y+e_z)+2\mu e_x \\ \sigma_y &= \lambda(e_x+e_y+e_z)+2\mu e_y \\ \sigma_z &= \lambda(e_x+e_y+e_z)+2\mu e_z \\ \tau_{xy} &= 2\mu e_{xy} \\ \tau_{yz} &= 2\mu e_{yz} \\ \tau_{zx} &= 2\mu e_{zx} \end{aligned} \quad \textbf{(4.8)}$$

Relations **(4.7)** or **(4.8)** are called the *generalized Hooke's law* for linear isotropic elastic solids. They are named after Robert Hooke who in 1678 first proposed that the deformation of an elastic structure is proportional to the applied force. Notice the significant simplicity of the isotropic form when compared to the general stress-strain law originally given by **(4.1).** It should be noted that only two independent elastic constants are needed to describe the behavior of isotropic materials. As shown, additional numbers of elastic moduli are needed in the corresponding relations for anisotropic materials.

Stress-strain relations **(4.7)** or **(4.8)** may be inverted to express the strain in terms of the stress. In order to do this it is convenient to use the index notation form (4.2.7) and set the two free indices the same (contraction process) to get

$$\sigma_{kk} = (3\lambda + 2\mu)e_{kk} \qquad \textbf{(4.9)}$$

This relation can be solved for e_{kk} and substituted back into **(4.7)** to get

$$e_{ij} = \frac{1}{2\mu}\left(\sigma_{ij} - \frac{\lambda}{3\lambda + 2\mu}\sigma_{kk}\delta_{ij}\right)$$

which is more commonly written as

$$e_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} \qquad \textbf{(4.10)}$$

where $E = \mu(3\lambda + 2\mu) = (\lambda + \mu)$ and is called the modulus of elasticity or *Young's modulus*, and $\nu = \lambda/[2(\lambda + \mu)]$ is referred to as *Poisson's ratio*. The index notation relation (4.10) may be written out in component (scalar) form giving the six equations

$$\begin{aligned}
e_x &= \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] \\
e_y &= \frac{1}{E}\left[\sigma_y - \nu(\sigma_z + \sigma_x)\right] \\
e_z &= \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right] \\
e_{xy} &= \frac{1+\nu}{E}\tau_{xy} = \frac{1}{2\mu}\tau_{xy} \\
e_{yz} &= \frac{1+\nu}{E}\tau_{yz} = \frac{1}{2\mu}\tau_{yz} \\
e_{zx} &= \frac{1+\nu}{E}\tau_{zx} = \frac{1}{2\mu}\tau_{zx}
\end{aligned} \qquad \textbf{(4.11)}$$

Constitutive form **(4.10)** or **(4.11)** again illustrates that only two elastic constants are needed to formulate *Hooke's law* for isotropic materials. By using any of the isotropic forms of Hooke's law, it can be shown that the principal axes of stress coincide with the principal axes of strain. This result also holds for some but not all anisotropic materials.

4.3. Physical Meaning of Elastic Moduli

For the isotropic case, the previously defined elastic moduli have simple physical meaning. These can be determined through investigation of particular states of stress commonly used in laboratory materials testing as shown in Figure 4-2.

4.3.1. Simple Tension

Consider the simple tension test as discussed previously with a sample subjected to tension in the x direction (see Figure 4-2). The state of stress is closely represented by the one-dimensional field

$$\sigma_{ij} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Using this in relations (**4.10**) gives a corresponding strain field

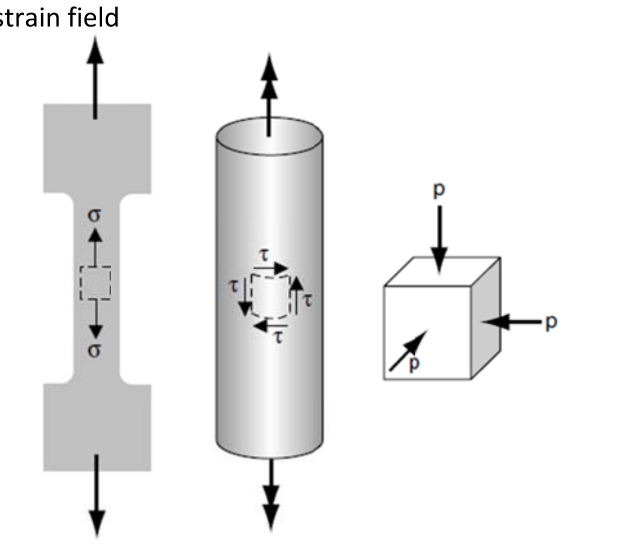

Figure 4-2 Special characterization states of stress.

$$e_{ij} = \begin{bmatrix} \frac{\sigma}{E} & 0 & 0 \\ 0 & \frac{-\nu\sigma}{E} & 0 \\ 0 & 0 & \frac{-\nu\sigma}{E} \end{bmatrix}$$

Therefore, $E = \sigma/e_x$ and is simply the slope of the stress-strain curve, while $v = -e_y/e_x = -e_z/e_x$ is the ratio of the transverse strain to the axial strain. Standard measurement systems can easily collect axial stress and transverse and axial strain data, and thus through this one type of test both elastic constants can be determined for materials of interest.

4.3.2. Pure Shear

If a thin-walled cylinder is subjected to torsional loading (as shown in Figure 4-2), the state of stress on the surface of the cylindrical sample is given by

$$\sigma_{ij} = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Again, by using *Hooke's law*, the corresponding strain field becomes

$$e_{ij} = \begin{bmatrix} 0 & \tau/2\mu & 0 \\ \tau/2\mu & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and thus the shear modulus is given by $\mu = \tau/2e_{xy} = \tau/\gamma_{xy}$, and this modulus is simply the slope of the shear stress-shear strain curve.

4.3.3. Hydrostatic Compression (or Tension)

The final example is associated with the uniform compression (or tension) loading of a cubical specimen, as shown in Figure 4-2. This type of test would be realizable if the sample was placed in a high-pressure compression chamber. The state of stress for this case is given by

$$\sigma_{ij} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} = -p\delta_{ij}$$

This is an isotropic state of stress and the strains follow from *Hooke's law*

$$e_{ij} = \begin{bmatrix} -\frac{1-2v}{E}p & 0 & 0 \\ 0 & -\frac{1-2v}{E}p & 0 \\ 0 & 0 & -\frac{1-2v}{E}p \end{bmatrix}$$

The dilatation that represents the change in material volume is thus given by $\vartheta = e_{kk} = -3(1-2v)p/E$, which can be written as

$$p = -k\,\vartheta$$

where $k = E/[3(1-2v)]$ is called the *bulk modulus of elasticity*. This additional elastic constant represents the ratio of pressure to the dilatation, which could be referred to as the volumetric stiffness of the material. Notice that as Poisson's ratio approaches 0.5, the bulk modulus becomes unbounded and the material does not undergo any volumetric deformation and hence is incompressible.

Our discussion of elastic moduli for isotropic materials has led to the definition of five constants λ, μ, E, v, and k. However, keep in mind that only two of these are needed to characterize the material. Although we have developed a few relationships between various moduli, many other such relations can also be found. In fact, it can be shown that all five elastic constants are interrelated, and if any two are given, the remaining three can be determined by using simple formulae. Results of these relations are conveniently summarized in Table 4-1. This table should be marked for future reference, because it will prove to be useful for calculations throughout the text.

Typical nominal values of elastic constants for particular engineering materials are given in Table 4-2. These moduli represent average values, and some variation will occur for specific materials.

TABLE 4-1 Relations Among Elastic Constants

	E	ν	k	μ	λ
E, ν	E	ν	$\frac{E}{3(1-2\nu)}$	$\frac{E}{2(1+\nu)}$	$\frac{E\nu}{(1+\nu)(1-2\nu)}$
E, k	E	$\frac{3k-E}{6k}$	k	$\frac{3kE}{9k-E}$	$\frac{3k(3k-E)}{9k-E}$
E, μ	E	$\frac{E-2\mu}{2\mu}$	$\frac{\mu E}{3(3\mu-E)}$	μ	$\frac{\mu(E-2\mu)}{3\mu-E}$
E, λ	E	$\frac{2\lambda}{E+\lambda+R}$	$\frac{E+3\lambda+R}{6}$	$\frac{E-3\lambda+R}{4}$	λ
ν, k	$3k(1-2\nu)$	ν	k	$\frac{3k(1-2\nu)}{2(1+\nu)}$	$\frac{3k\nu}{1+\nu}$
ν, μ	$2\mu(1+\nu)$	ν	$\frac{2\mu(1+\nu)}{3(1-2\nu)}$	μ	$\frac{2\mu\nu}{1-2\nu}$
ν, λ	$\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$	ν	$\frac{\lambda(1+\nu)}{3\nu}$	$\frac{\lambda(1-2\nu)}{2\nu}$	λ
k, μ	$\frac{9k\mu}{6k+\mu}$	$\frac{3k-2\mu}{6k+2\mu}$	k	μ	$k-\frac{2}{3}\mu$
k, λ	$\frac{9k(k-\lambda)}{3k-\lambda}$	$\frac{\lambda}{3k-\lambda}$	k	$\frac{3}{2}(k-\lambda)$	λ
μ, λ	$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	$\frac{\lambda}{2(\lambda+\mu)}$	$\frac{3\lambda+2\mu}{3}$	μ	λ

$R = \sqrt{E^2 + 9\lambda^2 + 2E\lambda}$

TABLE 4-2 Typical Values of Elastic Moduli for Common Engineering Materials

	E (GPa)	ν	μ(GPa)	λ(GPa)	k(GPa)	$\alpha(10^{-6}/°C)$
Aluminum	68.9	0.34	25.7	54.6	71.8	25.5
Concrete	27.6	0.20	11.5	7.7	15.3	11
Copper	89.6	0.34	33.4	71	93.3	18
Glass	68.9	0.25	27.6	27.6	45.9	8.8
Nylon	28.3	0.40	10.1	4.04	47.2	102
Rubber	0.0019	0.499	0.654×10^{-3}	0.326	0.326	200
Steel	207	0.29	80.2	111	164	13.5