Chapter 3- Deformation: Displacements and Strains

3.1. Introduction

We begin development of the basic field equations of elasticity theory by first investigating the kinematics of material deformation. As a result of applied loadings, elastic solids will change shape or deform, and these deformations can be quantified by knowing the displacements of material points in the body. The continuum hypothesis establishes a displacement field at all points within the elastic solid. Using appropriate geometry, particular measures of deformation can be constructed leading to the development of the strain tensor. As expected, the strain components are related to the displacement field. The purpose of this chapter is to introduce the basic definitions of displacement and strain, establish relations between these two field quantities, and finally investigate requirements to ensure single-valued, continuous displacement fields. As appropriate for linear elasticity, these kinematical results are developed under the conditions of small deformation theory. Developments in this chapter lead to two fundamental sets of field equations: the strain-displacement relations and the compatibility equations.

3.2. General Deformations

Under the application of external loading, elastic solids deform. A simple two-dimensional cantilever beam example is shown in Figure 3-1. The undeformed configuration is taken with the rectangular beam in the vertical position, and the end loading displaces material points to the deformed shape as shown. As is typical in most problems, the deformation varies from point to point and is thus said to be *nonhomogenous*. A superimposed square mesh is shown in the two configurations, and this indicates how elements within the material deform locally. It is apparent that elements within the mesh undergo extensional and shearing deformation. An elastic solid is said to be deformed or strained when the *relative displacements* between points in the body are changed. This is in contrast to *rigid-body* motion where the distance between points remains the same.

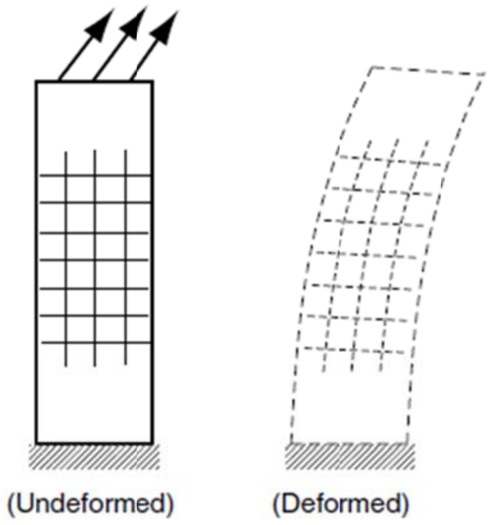

Figure 3-1 Two-dimensional deformation example.

In order to quantify deformation, consider the general example shown in Figure 3-2. In the undeformed configuration, we identify two neighboring material points P'_o and P connected with *the relative position vector* $\boldsymbol{r}$ as shown. Through a general deformation, these points are mapped to locations P'_o and P' in the deformed configuration. For *finite or large deformation theory*, the undeformed and deformed configurations can be significantly different, and a distinction between these two configurations must be maintained leading to Lagrangian and Eulerian descriptions. However, since we are developing linear elasticity, which uses only small deformation theory, the distinction between undeformed and deformed configurations can be dropped.

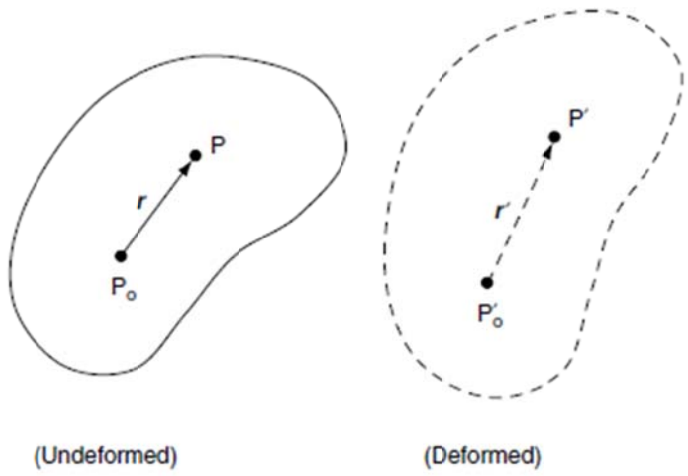

Figure 3-2 General deformation between two neighboring points.

Using Cartesian coordinates, define the displacement vectors of points Po and P to be $\boldsymbol{u}^o$ and $\boldsymbol{u}$, respectively. Since $\boldsymbol{P}$ and $\boldsymbol{P_o}$ are neighboring points, we can use a Taylor series expansion around point $\boldsymbol{P_o}$ to express the components of $\boldsymbol{u}$ as

$$\begin{aligned} u &= u^0 + \frac{\partial u}{\partial x} r_x + \frac{\partial u}{\partial y} r_y + \frac{\partial u}{\partial z} r_z \\ v &= v^0 + \frac{\partial v}{\partial x} r_x + \frac{\partial v}{\partial y} r_y + \frac{\partial v}{\partial z} r_z \\ w &= w^0 + \frac{\partial w}{\partial x} r_x + \frac{\partial w}{\partial y} r_y + \frac{\partial w}{\partial z} r_z \end{aligned} \qquad \textbf{(3.1)}$$

Note that the higher-order terms of the expansion have been dropped since the components of $\boldsymbol{r}$ are small. The change in the relative position vector $\boldsymbol{r}$ can be written as

$$\Delta \boldsymbol{r} = \boldsymbol{r}' - \boldsymbol{r} = \boldsymbol{u} - \boldsymbol{u}^o \qquad \textbf{(3.2)}$$

and using **(3.1)** gives

$$\begin{aligned} \Delta rx &= \frac{\partial u}{\partial x} r_x + \frac{\partial u}{\partial y} r_y + \frac{\partial u}{\partial z} r_z \\ \Delta ry &= \frac{\partial v}{\partial x} r_x + \frac{\partial v}{\partial y} r_y + \frac{\partial v}{\partial z} r_z \\ \Delta rz &= \frac{\partial w}{\partial x} r_x + \frac{\partial w}{\partial y} r_y + \frac{\partial w}{\partial z} r_z \end{aligned} \qquad \textbf{(3.3)}$$

or in index notation

$$\Delta r_i = u_{i,j}\, r_j \qquad \textbf{(3.4)}$$

The tensor $u_{i,j}$ is called the displacement gradient tensor, and may be written out as

$$u_{i,j} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} \qquad (3.5)$$

This tensor can be decomposed into symmetric and *antisymmetric* parts as

$$u_{i,j} = e_{ij}\, \omega_{ij} \qquad (3.6)$$

Where

$$\begin{aligned} e_{ij} &= \frac{1}{2}(u_{i,j} + u_{j,i}) \\ \omega_{ij} &= \frac{1}{2}(u_{i,j} - u_{j,i}) \end{aligned} \qquad (3.7)$$

The tensor e_{ij} is called the strain tensor, while ω_{ij} is referred to as the rotation tensor. Relations **(3.4)** and **(3.6)** thus imply that for small deformation theory, the change in the relative position vector between neighboring points can be expressed in terms of a sum of strain and rotation components. Combining relations **(3.2), (3.4),** and **(3.6),** and choosing $r_i = dx_i$, we can also write the general result in the form

$$u_i = u_i^o + e_{ij}\, dx_j + \omega_{ij} dx_j \qquad (3.8)$$

Because we are considering a general displacement field, these results include both strain deformation and rigid-body motion. That a dual vector ω_i can be associated with the rotation tensor such that $\omega_i = (-1/2)\varepsilon_{ijk}\, \omega_i$. Using this definition, it is found that

$$\begin{aligned} \omega_1 = \omega_{32} &= \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) \\ \omega_2 = \omega_{13} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ \omega_3 = \omega_{21} &= \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) \end{aligned} \qquad \textbf{(3.9)}$$

which can be expressed collectively in vector format as $\boldsymbol{\omega} = (1/2)(\nabla \times \boldsymbol{u})$. As is shown in the next section, these components represent rigid-body rotation of material elements about the coordinate axes. These general results indicate that the strain deformation is related to the strain tensor e_{ij}, which in turn is a related to the displacement gradients. We next pursue a more geometric approach and determine specific connections between the strain tensor components and geometric deformation of material elements.

3.3. Geometric Construction of Small Deformation Theory

Although the previous section developed general relations for small deformation theory, we now wish to establish a more geometrical interpretation of these results. Typically, elasticity variables and equations are field quantities defined at each point in the material continuum. However, particular field equations are often developed by first investigating the behavior of infinitesimal elements (with coordinate boundaries), and then a limiting process is invoked that allows the element to shrink to a point. Thus, consider the common deformational behavior of a rectangular element as shown in Figure 3-3. The usual types of motion include rigid-body rotation and extensional and shearing deformations as illustrated. Rigid-body motion does not contribute to the strain field, and thus also does not affect the stresses. We therefore focus our study primarily on the extensional and shearing deformation.

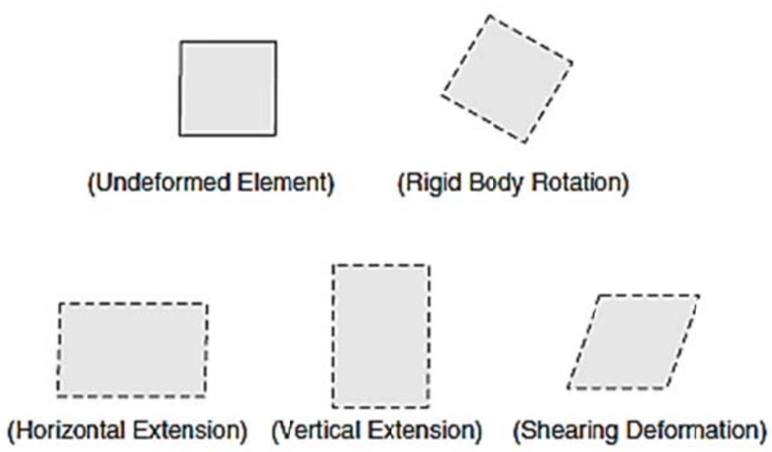

Figure 3-3 Typical deformations of a rectangular element.

Figure 3-4 illustrates the two-dimensional deformation of a rectangular element with original dimensions *dx* by *dy*. After deformation, the element takes a rhombus form as shown in the dotted outline. The displacements of various corner reference points are indicated in the figure. Reference point *A* is taken at location (x,y), and the displacement components of this point are *thus u(x,y)* and *v(x,y)*. The corresponding displacements of point B are *u(x + dx, y) and v(x + dx, y)*, and the displacements of the other corner points are defined in an analogous manner. According to small deformation theory, $u(x + dx, y) \approx u(x, y) + (\partial u/\partial x)\, dx$, with similar expansions for all other terms.

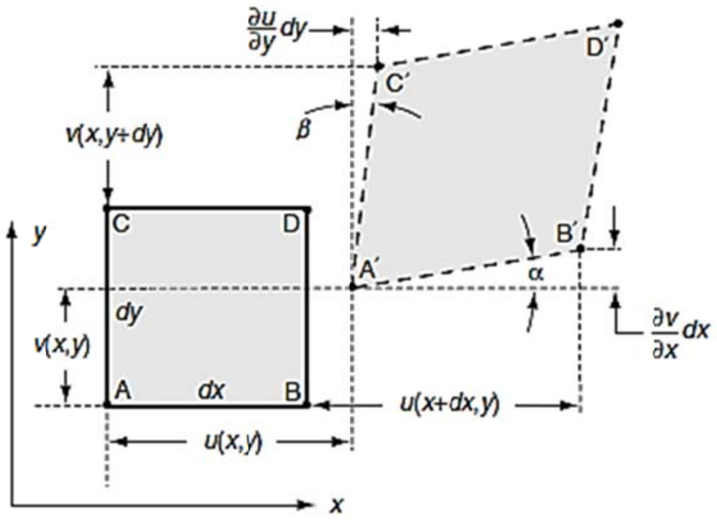

Figure 3-4 Two-dimensional geometric strain deformation.

The *normal or extensional strain component* in a direction n is defined as the change in length per unit length of fibers oriented in the n-direction. Normal strain is positive if fibers increase in length and negative if the fiber is shortened. In Figure 3-4, the normal strain in the x direction can thus be defined by

$$\varepsilon_x = \frac{A'B' - AB}{AB}$$

From the geometry in Figure 3-4,

$$A'B' = \sqrt{(dx + \frac{\partial u}{\partial x}dx)^2 + (\frac{\partial v}{\partial x}dx)^2} = \sqrt{1 + 2\frac{\partial u}{\partial x} + (\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial x})^2}\, dx \approx (1+\frac{\partial u}{\partial x})\, dx$$

where, consistent with small deformation theory, we have dropped the higher-order terms.

Using these results and the fact that *AB = dx*, the normal strain in the *x*-direction reduces to

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \mathbf{(3.10)}$$

In similar fashion, the normal strain in the *y*-direction becomes

$$\varepsilon_y = \frac{\partial v}{\partial y} \qquad \mathbf{(3.11)}$$

A second type of strain is shearing deformation, which involves angles changes (see Figure 3-3). *Shear strain* is defined as the change in angle between two originally orthogonal directions in the continuum material. This definition is actually referred to as the *engineering shear strain*. Theory of elasticity applications generally use a tensor formalism that requires a shear strain definition corresponding to one-half the angle change between orthogonal axes; see previous relation (3.7). Measured in radians, shear strain is positive if the right angle between the positive directions of the two axes decreases. Thus, the sign of the shear strain depends on the coordinate system. In Figure 3-4, the engineering shear strain with respect to the *x*- and *y*-directions can be defined as

$$\gamma_{xy} = \frac{\pi}{2} - \widehat{C'A'B'} = \alpha + \beta$$

For small deformations, α=tan α and β = tan β, and the shear strain can then be expressed as

$$\gamma_{xy} = \frac{\frac{\partial v}{\partial x}dx}{dx + \frac{\partial u}{\partial x}dx} + \frac{\frac{\partial u}{\partial y}dy}{dy + \frac{\partial v}{\partial y}dy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad \mathbf{(3.12)}$$

where we have again neglected higher-order terms in the displacement gradients. Note that each derivative term is positive if lines *AB* and *AC* rotate inward as shown in the figure. By simple interchange of *x* and *y* and u and *v*, it is apparent that $\gamma_{xy} = \gamma_{yx}$.

By considering similar behaviors in the *y-z* and *x-z* planes, these results can be easily extended to the general three-dimensional case, giving the results:

$$\begin{aligned} &\varepsilon_x = \frac{\partial u}{\partial x},\ \varepsilon_y = \frac{\partial v}{\partial y},\ \varepsilon_z = \frac{\partial w}{\partial z} \\ &\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},\ \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y},\ \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \end{aligned} \qquad \mathbf{(3.13)}$$

Thus, we define three normal and three shearing strain components leading to a total of six independent components that completely describe small deformation theory. This set of equations is normally referred to as the *strain-displacement relations*. However, these results are written in terms of the engineering strain components, and tensorial elasticity theory prefers to use the strain tensor eij defined by **(3.7)**. This represents only a minor change because the normal strains are identical and shearing strains differ by a factor of one-half; for example, $e_{11} = e_x = ex$ and $e_{12} = e_{xy} = (1/2)\,\gamma_{xy}$, and so forth.

Therefore, using the strain tensor e_{ij}, the strain-displacement relations can be expressed in component form as

$$e_x = \frac{\partial u}{\partial x}, e_y = \frac{\partial v}{\partial y}, \ e_z = \frac{\partial w}{\partial z}$$
$$e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \ e_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \ e_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \quad \textbf{(3.14)}$$

Using the more compact tensor notation, these relations are written as

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad \textbf{(3.15)}$$

while in direct vector/matrix notation as the form reads:

$$\boldsymbol{e} = \frac{1}{2}\left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right] \quad \textbf{(3.16)}$$

where e is the strain matrix and $\nabla \boldsymbol{u}$ is the displacement gradient matrix and $(\nabla \boldsymbol{u})^T$ is its transpose.

The strain is a symmetric second-order tensor ($e_{ij} = e_{ji}$) and is commonly written in matrix format:

$$\boldsymbol{e} = [\boldsymbol{e}] = \begin{bmatrix} e_x & e_{xy} & e_{xz} \\ e_{xy} & e_y & e_{yz} \\ e_{xz} & e_{yz} & e_z \end{bmatrix} \quad \textbf{(3.17)}$$

Before we conclude this geometric presentation, consider the rigid-body rotation of our two-dimensional element in the *x-y* plane, as shown in Figure 3-5. If the element is rotated through a small rigid-body angular displacement about the *z*-axis, using the bottom element edge, the rotation angle is determined as$\partial v/\partial x$, while using the left edge, the angle is given by $-\partial u/\partial y$. These two expressions are of course the same; that is, $\partial v/\partial x = -\partial u/\partial y$ and note that this would imply $e_{xy} = 0$. The rotation can then be expressed as $\omega_z = [(\partial v/\partial x) - (\partial u/\partial y)]/2$, which matches with the expression given earlier in (3. 9). The other components of rotation follow in an analogous manner.

Relations for the constant rotation ω_z can be integrated to give the result:

$$u^* = u_o - \omega_z y$$
$$v^* = v_o + \omega_z x \quad \textbf{(3.18)}$$

where u_o and v_o are arbitrary constant translations in the *x*- and *y*-directions. This result then specifies the general form of the displacement field for two-dimensional rigid-body motion. We can easily verify that the displacement field given by **(3.18)** yields zero strain.

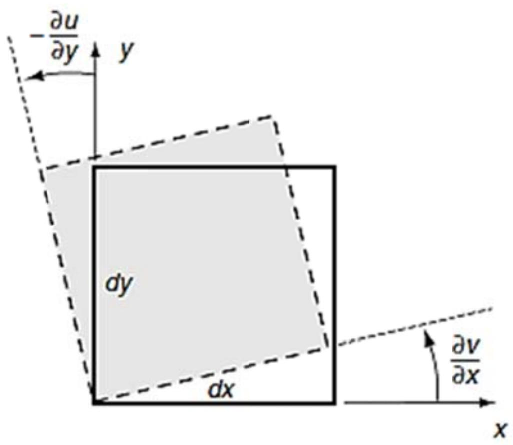

Figure 3-5 Two-dimensional rigid-body rotation.

For the three-dimensional case, the most general form of rigid-body displacement can be expressed as

$$u^* = u_o - \omega_z y + \omega_y z$$
$$v^* = v_o - \omega_x z + \omega_z x \quad \textbf{(3.19)}$$
$$w^* = w_o - \omega_y x + \omega_x y$$

As shown later, integrating the strain-displacement relations to determine the displacement field produces arbitrary constants and functions of integration, which are equivalent to rigid-body motion terms of the form given by (**3.18**) or (**3.19**). Thus, it is important to recognize such terms because we normally want to drop them from the analysis since they do not contribute to the strain or stress fields.

3.4. Strain Transformation

Because the strains are components of a second-order tensor, the transformation theory discussed in Section 1can be applied. Transformation relation **(1.1)** is applicable for second-order tensors, and applying this to the strain gives

$$e'_{ij} = Q_{ip}\, Q_{jp}\, e_{pq} \quad \textbf{(3. 20)}$$

where the rotation matrix $Q_{ij} = \cos(x'_i, x_j)$. Thus, given the strain in one coordinate system, we can determine the new components in any other rotated system. For the general three-dimensional case, define the rotation matrix as

$$Q_{ij} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \quad \textbf{(3.21)}$$

Using this notational scheme, the specific transformation relations from equation **(3.20)** become

$$\begin{aligned}
e'_x &= e_x l_1^2 + e_y m_1^2 + e_z n_1^2 + 2(e_{xy} l_1 m_1 + e_{yz} m_1 n_1 + e_{zx} n_1 l_1) \\
e'_y &= e_x l_2^2 + e_y m_2^2 + e_z n_2^2 + 2(e_{xy} l_2 m_2 + e_{yz} m_2 n_2 + e_{zx} n_2 l_2) \\
e'_z &= e_x l_3^2 + e_y m_3^2 + e_z n_3^2 + 2(e_{xy} l_3 m_3 + e_{yz} m_3 n_3 + e_{zx} n_3 l_3) \\
e'_{xy} &= e_x l_1 l_2 + e_y m_1 m_2 + e_z n_1 n_2 + e_{xy}(l_1 m_2 + m_1 l_2) + e_{yz}(m_1 n_2 + n_1 m_2) + e_{zx}(n_1 l_2 + l_1 n_2) \\
e'_{yz} &= e_x l_2 l_3 + e_y m_2 m_3 + e_z n_2 n_3 + e_{xy}(l_2 m_3 + m_2 l_3) + e_{yz}(m_2 n_3 + n_2 m_3) + e_{zx}(n_2 l_3 + l_2 n_3) \\
e'_{zx} &= e_x l_3 l_1 + e_y m_3 m_1 + e_z n_3 n_1 + e_{xy}(l_3 m_1 + m_3 l_1) + e_{yz}(m_3 n_1 + n_3 m_1) + e_{zx}(n_3 l_1 + l_3 n_1)
\end{aligned} \qquad \textbf{(3.22)}$$

For the two-dimensional case shown in Figure 3-6, the transformation matrix can be expressed as

$$Q_{ij} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \textbf{(3.23)}$$

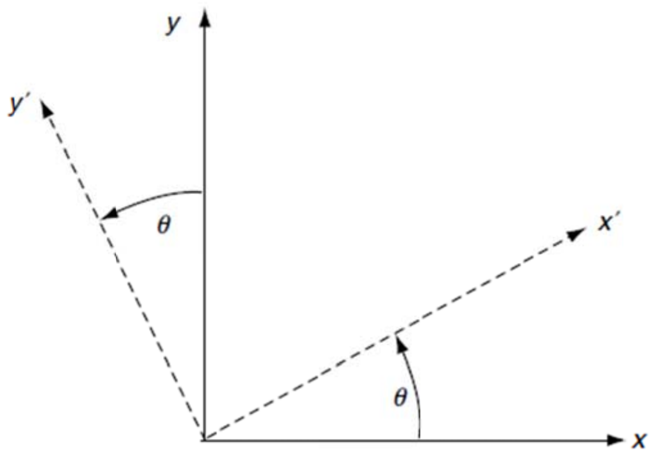

Figure 3-5 Two-dimensional rotational transformation.

Under this transformation, the in-plane strain components transform according to

$$\begin{aligned}
e'_x &= e_x \cos^2\theta + e_y \sin^2\theta + 2e_{xy}\sin\theta\cos\theta \\
e'_y &= e_x \sin^2\theta + e_y \cos^2\theta - 2e_{xy}\sin\theta\cos\theta \\
e'_{xy} &= -e_x \sin\theta\cos\theta + e_y \sin\theta\cos\theta + e_{xy}(\cos^2\theta - \sin^2\theta)
\end{aligned} \qquad \textbf{(3.24)}$$

which is commonly rewritten in terms of the double angle:

$$\begin{aligned}
e'_x &= \frac{e_x + e_y}{2} + \frac{e_x - e_y}{2}\cos 2\theta + e_{xy}\sin 2\theta \\
e'_y &= \frac{e_x + e_y}{2} - \frac{e_x - e_y}{2}\cos 2\theta - e_{xy}\sin 2\theta \\
e'_{xy} &= \frac{e_y - e_x}{2}\sin 2\theta + e_{xy}\cos 2\theta
\end{aligned} \qquad \textbf{(3.25)}$$

Transformation relations (**3.25**) can be directly applied to establish transformations between Cartesian and polar coordinate systems. Additional applications of these results can be found when dealing with experimental strain gage measurement systems. For example, standard experimental methods using a rosette strain gage allow the determination of extensional strains in three different directions on the surface of a structure. Using this type of data, relation (**3.25**) can be repeatedly used to establish three independent equations that can be solved for the state of strain (e_x, e_y, e_{xy}) at the surface point under study.

3.5. Principal Strains

From the previous discussion ,it follows that because the strain is a symmetric second-order tensor, we can identify and determine its principal axes and values. According to this theory, for any given strain tensor we can establish the principal value problem and solve the characteristic equation to explicitly determine the principal values and directions. The general characteristic equation for the strain tensor can be written as

$$\det[e_{ij} - e\delta_{ij}] = -e^3 + \vartheta_1 e^2 - \vartheta_2 e + \vartheta_3 = 0 \qquad \textbf{(3.26)}$$

where e is the principal strain and the fundamental invariants of the strain tensor can be expressed in terms of the three principal strains e_1, e_2, e_3 as

$$\begin{aligned} \vartheta_1 &= e_1 + e_2 + e_3 \\ \vartheta_2 &= e_1 e_2 + e_2 e_3 + e_3 e_1 \\ \vartheta_3 &= e_1 e_2 e_3 \end{aligned} \tag{3.27}$$

The first invariant $\vartheta_1 = \vartheta$ is normally called the *cubical dilatation*, because it is related to the change in volume of material elements.

The strain matrix in the principal coordinate system takes the special diagonal form

$$e_{ij} = \begin{bmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{bmatrix} \tag{3.28}$$

Notice that for this principal coordinate system, the deformation does not produce any shearing and thus is only extensional. Therefore, a rectangular element oriented along principal axes of strain will retain its orthogonal shape and undergo only extensional deformation of its sides.