Chapter 2- Stress tensor and Equilibrium

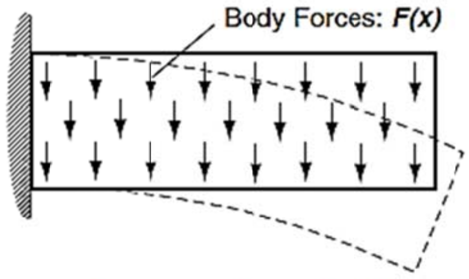
2.1. Introduction

The previous chapter investigated the kinematics of deformation without regard to the force or stress distribution within the elastic solid. We now wish to examine these issues and explore the transmission of forces through deformable materials. Our study leads to the definition and use of the traction vector and stress tensor. Each provides a quantitative method to describe both boundary and internal force distributions within a continuum solid. Because it is commonly accepted that maximum stresses are a major contributing factor to material failure, primary application of elasticity theory is used to determine the distribution of stress within a given structure. Related to these force distribution issues is the concept of equilibrium. Within a deformable solid, the force distribution at each point must be balanced. For the static case, the summation of forces on an infinitesimal element is required to be zero, while for a dynamic problem the resultant force must equal the mass times the element's acceleration. In this chapter, we establish the definitions and properties of the traction vector and stress tensor and develop the equilibrium equations, which become another set of field equations necessary in the overall formulation of elasticity theory. It should be noted that the developments in this chapter do not require that the material be elastic, and thus in principle these results apply to a broader class of material behavior.

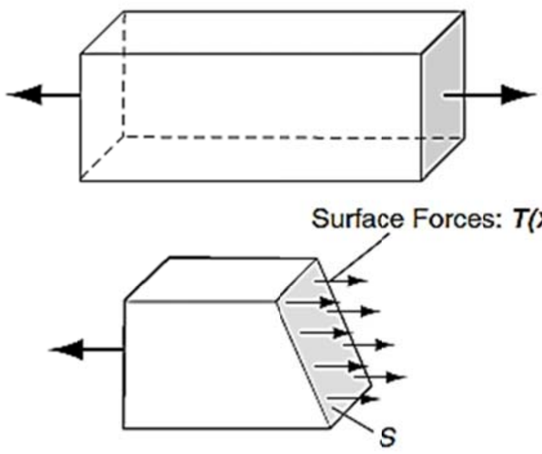
2.2. Body and Surface Forces

When a structure is subjected to applied external loadings, internal forces are induced inside the body. Following the philosophy of continuum mechanics, these internal forces are distributed continuously within the solid. In order to study such forces, it is convenient to categorize them into two major groups, commonly referred to as body forces and surface forces.

Body forces are proportional to the body's mass and are reacted with an agent outside of the body. Examples of these include gravitational-weight forces, magnetic forces, and inertial forces. Figure 2-1(a) shows an example body force of an object's self-weight. By using continuum mechanics principles, a ***body force density*** (force per unit volume) ***F(x)*** can be defined such that the total resultant body force of an entire solid can be written as a volume integral over the body



(a) Cantilever Beam Under Self-Weight Loading



(b) Sectioned Axially Loaded Beam

Figure 2-1 Examples of body and surface forces.

$$F_R = \iiint_V F(x)dV$$

Surface forces always act on a surface and result from physical contact with another body. Figure 2-1(b) illustrates surface forces existing in a beam section that has been created by sectioning the body into two pieces. For this particular case, the surface S is a virtual one in the sense that it was artificially created to investigate the nature of the internal forces at this location in the body. Again the resultant surface force over the entire surface S can be expressed as the integral of a surface force density function $\mathbf{T^n(x)}$

$$F_S = \iint_S T^n(x)dS$$

The surface force density is normally referred to as the traction vector and is discussed in more detail in the next section. In the development of classical elasticity, distributions of body or surface couples are normally not included. Theories that consider such force distributions have been constructed in an effort to extend classical elasticity for applications in micromechanical modeling. Such approaches are normally called *micropolar* or *couplestress theory*

2.3. Traction Vector and Stress Tensor

In order to quantify the nature of the internal distribution of forces within a continuum solid, consider a general body subject to arbitrary (concentrated and distributed) external loadings, as shown in Figure 2-2. To investigate the internal forces, a section is made through the body as shown. On this section consider a small area ΔA with unit normal vector **n**. The resultant surface force acting on ΔA is defined by ΔF. Consistent with our earlier discussion; no resultant surface couple is included. The stress or traction vector is defined by

$$T^n(x,n) = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} \qquad (2.1)$$

Notice that the traction vector depends on both the spatial location and the unit normal vector to the surface under study. Thus, even though we may be investigating the same point, the traction vector still varies as a function of the orientation of the surface normal. Because the traction is defined as force per unit area, the total surface force is determined through integration as per relation (**1**). Note, also, the simple action-reaction principle (Newton's third law)

$$T^n(x,n) = T^n(x,-n)$$

Consider now the special case in which DA coincides with each of the three coordinate planes with the unit normal vectors pointing along the positive coordinate axes. This concept is shown in Figure 2-3, where the three coordinate surfaces for ΔA partition off a cube of material. For this case, the traction vector on each face can be written as

$$\begin{aligned} T^n(x,n=e1) &= \sigma x\, e1 + \tau xy\, e2 + \tau xz e3 \\ T^n(x,n=e2) &= \tau yx\, e1 + \sigma y\, e2 + \tau yz e3 \\ T^n(x,n=e3) &= \tau zx\, e1 + \tau zy e2 + \sigma z\, e3 \end{aligned} \qquad \mathbf{(2.2)}$$

where e1, e2, e3 are the unit vectors along each coordinate direction, and the nine quantities {σx, σy, σz, τxy, τyx, τyz, τzy, τzx, τxz} are the components of the traction vector on each of three coordinate planes as illustrated. These nine components are called the *stress components*,

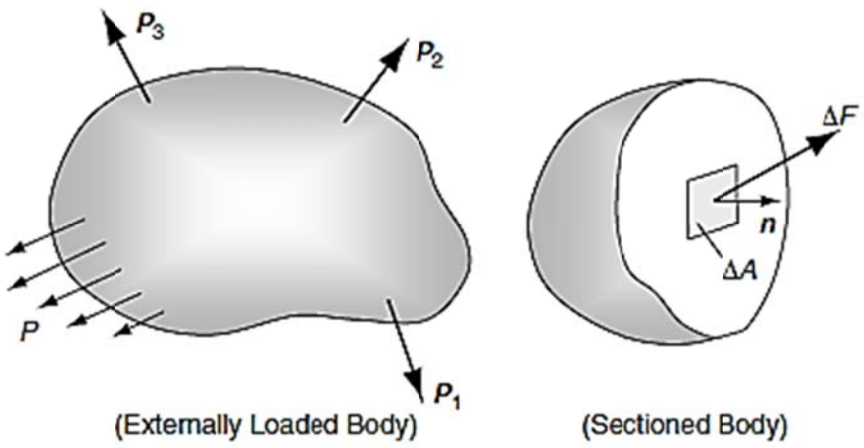


Figure 2-2 Sectioned solid under external loading.

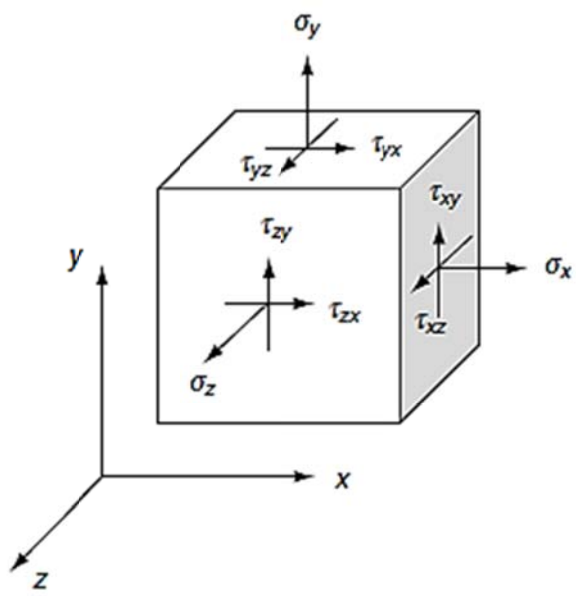


Figure 2-3 Components of the stress.

with σx, σy, σz referred to as *normal stresses* and τ_{xy}, τ_{yx}, τ_{yz}, τ_{zy}, τ_{zx}, τ_{xz} called the *shearing stresses*. The components of stress σij are commonly written in matrix format

$$\sigma = [\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

and it can be formally shown that the stress is a second-order tensor that obeys the appropriate transformation law

The positive directions of each stress component are illustrated in Figure 2-3. Regardless of the coordinate system, positive normal stress always acts in tension out of the face, and only one subscript is necessary because it always acts normal to the surface. The shear stress, however, requires two subscripts, the first representing the plane of action and the second designating the direction of the stress. Similar to shear strain, the sign of the shear stress depends on coordinate system orientation. For example, on a plane with a normal in the positive x direction, positive τxy acts in the positive y direction. Similar definitions follow for the other shear stress components. In subsequent chapters, proper formulation of elasticity problems requires knowledge of these basic definitions, directions, and sign conventions for particular stress components.

Consider next the traction vector on an oblique plane with arbitrary orientation, as shown in Figure 2-4. The unit normal to the surface can be expressed by

$$\boldsymbol{n} = n_x\, e_1 + n_y\, e_2 + n_z e_3 \qquad \mathbf{(2.3)}$$

where n_x, n_y, n_z are the direction cosines of the unit vector $\boldsymbol{n}$ relative to the given coordinate system. We now consider the equilibrium of the pyramidal element interior to the oblique and coordinate planes. Invoking the force balance between tractions on the oblique and coordinate faces gives

$$\boldsymbol{T}^n = n_x \boldsymbol{T}^n(n = e1) + n_y \boldsymbol{T}^n(n = e2) + n_z \boldsymbol{T}^n(n = e3)$$

and by using relations (**2**), this can be written as

$$\boldsymbol{T}^n = (\sigma_x n_x + \tau_{yx} n_y + \tau_{zx} n_z)e1 + (\tau_{xy} n_x + \sigma y n_y + \tau_{zy} n_z)e2 + (\tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z)e3 \qquad \textbf{(2.4)}$$

or in index notation

$$\boldsymbol{T}^n_i = \sigma_{ji} n_j \qquad \textbf{(2.5)}$$

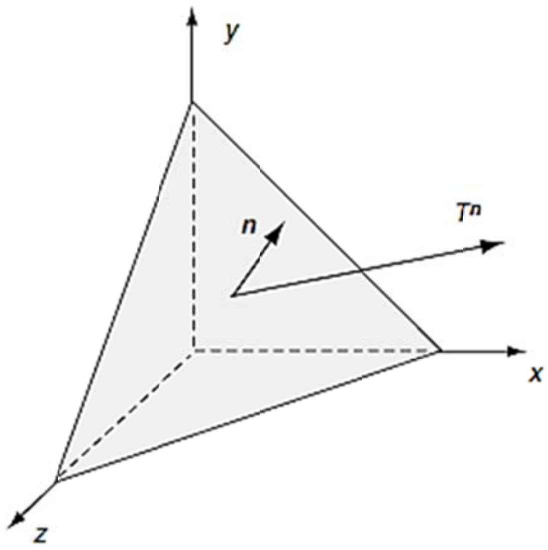


Figure 2-4 Traction on an oblique plane.

Relation (**2.4**) or (**2.5**) provides a simple and direct method to calculate the forces on oblique planes and surfaces. This technique proves to be very useful to specify general boundary conditions during the formulation and solution of elasticity problems.

Following the principles of small deformation theory, the previous definitions for the stress tensor and traction vector do not make a distinction between the deformed and undeformed configurations of the body. As mentioned in the previous chapter, such a distinction only leads to small modifications that are considered ***higher-order effects*** and are normally neglected. However, for large deformation theory, sizeable differences exist between these configurations, and the undeformed configuration (commonly called the reference configuration) is often used in problem formulation. This gives rise to the definition of an additional stress called the ***Piola-Kirchhoff stress tensor*** that represents the force per unit area in the reference configuration). In the more general scheme, the stress σij is referred to as the Cauchy stress tensor. Throughout the text only small deformation theory is considered, and thus the distinction between these two definitions of stress disappears, thereby eliminating any need for this additional terminology.

2.4. Stress Transformation

Analogous to our previous discussion with the strain tensor, the stress components must also follow the standard transformation rules for second-order tensors. Applying transformation relation (**1.1**) for the stress gives

$$\sigma'_{ij} = Q_{ip} Q_{jq} \sigma_{pq}$$

where the rotation matrix Q_{ij} + cos (x'i, xj). Therefore, given the stress in one coordinate system, we can determine the new components in any other rotated system. For the general three-dimensional case, the rotation matrix may be chosen in the form

$$Q_{ij} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$$

Using this notational scheme, the specific transformation relations for the stress then become

$$\begin{aligned}
\sigma'x &= \sigma_x l^2_1 + \sigma y\, m^2_1 + \sigma_z n^2_1 + 2(\tau_{xy} l_1 m_1 + \tau_{yz} m_1 n_1 + \tau_{zx} n_1 l_1) \\
\sigma'y &= \sigma_x l^2_2 + \sigma y\, m^2_2 + \sigma_z n^2_2 + 2(\tau_{xy} l_2 m_2 + \tau_{yz} m_2 n_2 + \tau_{zx} n_2 l_2) \\
\sigma'z &= \sigma_x l^2_3 + \sigma y\, m^2_3 + \sigma_z n^2_3 + 2(\tau_{xy} l_3 m_3 + \tau_{yz} m_3 n_3 + \tau_{zx} n_3 l_3) \\
\tau'_{xy} &= \sigma_x l_1 l_2 + \sigma_y m_1 m_2 + \sigma_z n_1 n_2 + \tau_{xy}(l_1 m_2 + m_1 l_2) + \tau_{yz}(m_1 n_2 + n_1 m_2) + \tau_{zx}(n_1 l_2 + l_1 n_2) \\
\tau'_{yz} &= \sigma_x l_2 l_3 + \sigma_y m_2 m_3 + \sigma_z n_2 n_3 + \tau_{xy}(l_2 m_3 + m_2 l_3) + \tau_{yz}(m_2 n_3 + n_2 m_3) + \tau_{zx}(n_2 l_3 + l_2 n_3) \\
\tau'_{zx} &= \sigma_x l_3 l_1 + \sigma_y m_3 m_1 + \sigma_z n_3 n_1 + \tau_{xy}(l_3 m_1 + m_3 l_1) + \tau_{yz}(m_3 n_1 + n_3 m_1) + \tau_{zx}(n_3 l_1 + l_1 n_1)
\end{aligned} \qquad \textbf{(2.6)}$$

For the two-dimensional case originally shown in Figure 2-5, the transformation matrix was given by relation (**2.7**).

$$Q_{ij} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \textbf{(2.7)}$$

Under this transformation, the in-plane stress components transform according to

$$\begin{aligned}
\sigma'_x &= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta\cos\theta \\
\sigma'_y &= \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy} \sin\theta\cos\theta \\
\tau'_{xy} &= -\sigma_x \sin\theta\cos\theta + \sigma_y \sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)
\end{aligned}$$

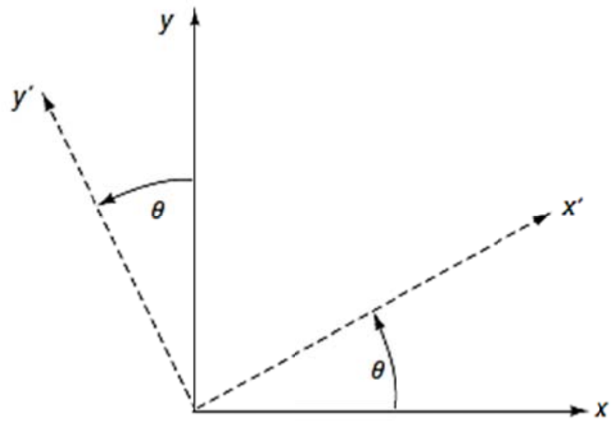


Figure 2-5 Two-dimensional rotational transformation.

which is commonly rewritten in terms of the double angle

$$\sigma x' = \frac{\sigma x + \sigma y}{2} + \frac{\sigma x - \sigma y}{2}\cos 2\theta + \tau xy \sin 2\theta$$
$$\sigma y' = \frac{\sigma x + \sigma y}{2} - \frac{\sigma x - \sigma y}{2}\cos 2\theta - \tau xy \sin 2\theta \qquad \textbf{(2.8)}$$
$$\sigma' = \frac{\sigma y - \sigma x}{2}\sin 2\theta + \tau xy \cos 2\theta$$

Similar to our discussion on strain in the previous chapter, relations **(2.8)** can be directly applied to establish stress transformations between Cartesian and polar coordinate systems. Both two- and three-dimensional stress transformation equations can be easily incorporated in MATLAB to provide numerical solution to problems of interest

2.5. Principal Stresses

We can use the developments to discuss the issues of principal stresses and directions. It is shown later in the chapter that the stress is a symmetric tensor. Using this fact, appropriate theory has been developed to identify and determine principal axes and values for the stress. For any given stress tensor we can establish the principal value problem and solve the characteristic equation to explicitly determine the principal values and directions. The general characteristic equation for the stress tensor becomes

$$\det[\sigma_{ij} - \sigma\,\delta_{ij}] = -\sigma_3 + I_1\sigma^2 - I_2\,\sigma + I_3 = 0$$

where σ are the principal stresses and the fundamental invariants of the stress tensor can be expressed in terms of the three principal stresses σ_1, σ_2, σ_3 as

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$
$$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$
$$I_3 = \sigma_1\sigma_2\sigma_3$$

In the principal coordinate system, the stress matrix takes the special diagonal form

$$\sigma ij = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

A comparison of the general and principal stress states is shown in Figure 2-6. Notice that for the principal coordinate system, all shearing stresses vanish and thus the state includes only normal stresses.

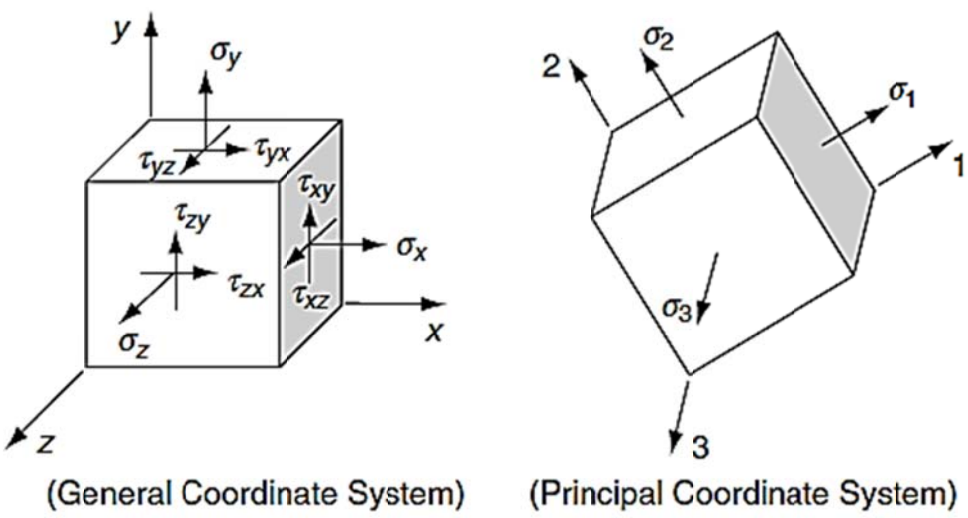


Figure 2-6 Comparison of general and principal stress states.

We now wish to go back to investigate another issue related to stress and traction transformation that makes use of principal stresses. Consider the general traction vector $\boldsymbol{T}^n$ that acts on an arbitrary surface as shown in Figure 2-7. The issue of interest is to determine the traction vector's normal and shear components *N* and *S*. The normal component is simply the traction's projection in the direction of the unit normal vector $\boldsymbol{n}$, while the shear component is found by Pythagorean theorem,

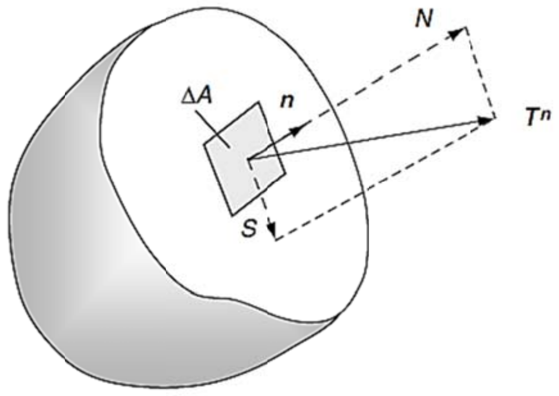


Figure 2-7 Traction vector decomposition.

$$\boldsymbol{N} = \boldsymbol{T^n}.\boldsymbol{n} \quad \textbf{(2.9)}$$

$$S = (|T^n|^2 - N^2)^{1/2}$$

Using the relationship for the traction vector **(2.5)** into **(2.9)** gives

$$N = T^n.n = T_i^n.n_i = \sigma_{ji}\, n_j\, n_i$$

$$= \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2$$

where, in order to simplify the expressions, we have used the principal axes for the stress tensor. In a similar manner,

$$|\boldsymbol{T^n}|^2 = \boldsymbol{T^n}.\boldsymbol{T^n} = T_i^n.T_i^n = \sigma_{ji} n_j \sigma_{ki} n_k$$
$$= \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2$$

Using these results back into relation **(2.9**) yields

$$N = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 \quad \textbf{(2.10)}$$
$$S^2 + N^2 = \sigma_1^2 n12 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2$$

In addition, we also add the condition that the vector ***n*** has unit magnitude

$$1 = n_1^2 + n_2^2 + n_3^2 \quad \textbf{(2.11)}$$

Relations (3.4.7) and (3.4.8) can be viewed as three linear algebraic equations for the unknowns n_1, $n2^2$, n_3^2 Solving this system gives the following result:

$$n_1^2 = \frac{S^2 + (N - \sigma_2)(N - \sigma_3)}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)}$$
$$n_2^2 = \frac{S^2 + (N - \sigma_3)(N - \sigma_1)}{(\sigma_2 - \sigma_3)(\sigma_2 - \sigma_1)}$$
$$n_3^2 = \frac{S^2 + (N - \sigma_1)(N - \sigma_2)}{(\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)} \quad \textbf{(2.12)}$$

Without loss in generality, we can rank the principal stresses as $\sigma_1 > \sigma_2 > \sigma_3$. Noting that the expressions given by **(2.12)**must be greater than or equal to zero, we can conclude the following

$$S^2 + (N-\sigma_2)(N-\sigma_3) \geq 0$$
$$S^2 + (N-\sigma_3)(N-\sigma_1) \geq 0 \quad \textbf{(2.13)}$$
$$S^2 + (N-\sigma_1)(N-\sigma_2) \geq 0$$

For the equality case, equations **(2.13)** represent three circles in an S-N coordinate system, and Figure 3-7 illustrates the location of each circle. These results were originally generated by **Otto Mohr** over a century ago, and the circles are commonly called Mohr's circles of stress.

The three inequalities given in (3.4.10) imply that all admissible values of N and S lie in the shaded regions bounded by the three circles. Note that, for the ranked principal stresses, the largest shear component is easily determined as $S_{max} = 1/2\ |\sigma_1 - \sigma_3|$. Although these circles can be effectively used for two-dimensional stress transformation, the general tensorial-based equations **(2.6)** are normally used for general transformation computations.

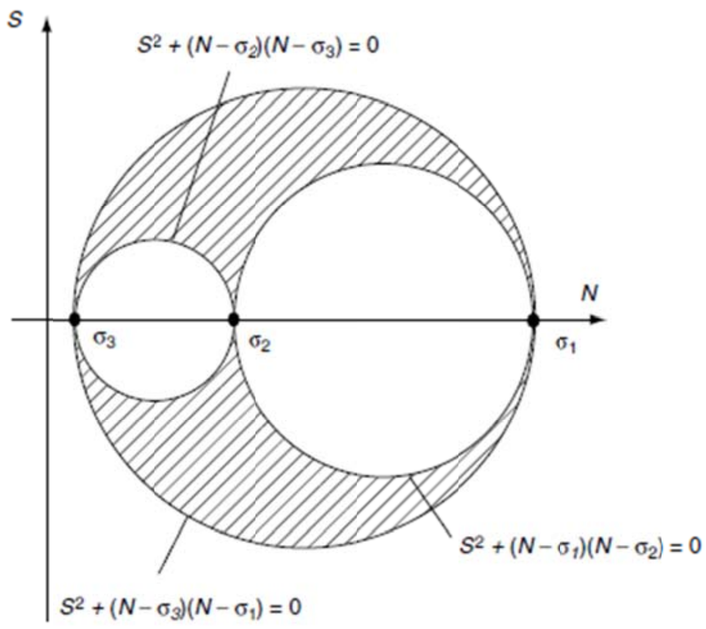


Figure 2-8 Mohr's circles of stress.