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Chapter 1- **Foundations and Elementary Applications**

1.1. Mathematical Preliminaries

Similar to other field theories such as fluid mechanics, heat conduction, and electromagnetics, the study and application of elasticity theory requires knowledge of several areas of applied mathematics. The theory is formulated in terms of a variety of variables including scalar, vector, and tensor fields, and these calls for the use of tensor notation along with tensor algebra and calculus. Through the use of particular principles from continuum mechanics, the theory is developed as a system of partial differential field equations that are to be solved in a region of space coinciding with the body under study. Solution techniques used on these field equations commonly employ Fourier methods, variational techniques, integral transforms, complex variables, potential theory, finite differences, and finite and boundary elements. Therefore, to develop proper formulation methods and solution techniques for elasticity problems, it is necessary to have an appropriate mathematical background. The purpose of this initial chapter is to provide a background primarily for the formulation part of our study. Additional review of other mathematical topics related to problem solution technique is provided in later chapters where they are to be applied.

1.1.1. Scalar, Vector, Matrix, and Tensor Definitions

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Elasticity theory is formulated in terms of many different types of variables that are either specified or sought at spatial points in the body under study. Some of these variables are:

A- Scalar quantities: representing a single magnitude at each point in space. Common examples include the material density ρ and material moduli such as Young's modulus E, Poisson's ratio υ, or the shear modulus μ.

B- Vector quantities: Those are expressible in terms of components in a two- or three-dimensional coordinate system. Examples of vector variables are the displacement and rotation of material points in the elastic continuum.

C- Formulations within the theory also require the need for *matrix variables*, which commonly require more than three components to quantify. Examples of such variables include stress and strain.

As shown above, a three-dimensional formulation requires nine components (only six are independent) to quantify the stress or strain at a point. For this case, the variable is normally expressed in a matrix format with three rows and three columns.

To summarize this discussion, in a three-dimensional Cartesian coordinate system, scalar, vector, and matrix variables can thus be written as follows:

Mass density scalar = ρ

Displacement vector = u = $ue_1 + ve_2 + we_3$

$$\text{Stress matrix} = [\sigma] = \begin{bmatrix} \sigma x & \tau xt & \tau xz \\ \tau yx & \sigma y & \tau yz \\ \tau zx & zy & \sigma z \end{bmatrix}$$

where e_1, e_2, e_3 are the usual unit basis vectors in the coordinate directions. Thus, scalars, vectors, and matrices are specified by one, three, and nine components, respectively.

The formulation of elasticity problems not only involves these types of variables, but also incorporates additional quantities that require even more components to characterize. Because of this, most field theories such as elasticity make use of a tensor formalism using index notation.

This enables efficient representation of all variables and governing equations using a single standardized scheme. The tensor concept is defined more precisely in a later section, but for now we can simply say that scalars, vectors, matrices, and other higher-order variables can all be represented by tensors of various orders. We now proceed to a discussion on the notational rules of order for the tensor formalism.

1.1.2. Index Notation

Index notation is a shorthand scheme whereby a whole set of numbers (elements or components) is represented by a single symbol with subscripts. For example, the three numbers a_1, a_2, a_3 are denoted by the symbol a_i, where index i will normally have the range 1, 2, 3. In a similar fashion, a_{ij} represents the nine numbers a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a33.

Although these representations can be written in any manner, it is common to use a scheme related to vector and matrix formats such that

$$ai = \begin{bmatrix} a1 \\ a2 \\ a3 \end{bmatrix}, \qquad aij = \begin{bmatrix} a11 & a12 & a13 \\ a21 & a22 & a13 \\ a31 & a32 & a33 \end{bmatrix}$$

In the matrix format, a_{1j} represents the first row, while a_{i1} indicates the first column. Other columns and rows are indicated in similar fashion, and thus the first index represents the row, while the second index denotes the column.

In general a symbol $a_{ij\ldots k}$ with *N* distinct indices represents 3^N distinct numbers. It should be apparent that a_i and a_j represent the same three numbers, and likewise a_{ij} and a_{mn} signify the same matrix. Addition, subtraction, multiplication, and equality of index symbols are defined in the normal fashion. For example, addition and subtraction are given by

$$ai \pm bi = \begin{bmatrix} a1 \pm b1 \\ a2 \pm b2 \\ a3 \pm b3 \end{bmatrix}, \quad aij \pm bij = \begin{bmatrix} a11 \pm b11 & a12 \pm b12 & a13 \pm b13 \\ a21 \pm b21 & a22 \pm b22 & a23 \pm b23 \\ a31 \pm b31 & a32 \pm b32 & a33 \pm b33 \end{bmatrix}$$

and scalar multiplication is specified as

$$\lambda ai = \begin{bmatrix} \lambda a1 \\ \lambda a2 \\ \lambda a3 \end{bmatrix}, \qquad \lambda aij = \begin{bmatrix} \lambda a11 & \lambda a12 & \lambda a13 \\ \lambda a21 & \lambda a22 & \lambda a13 \\ \lambda a31 & \lambda a32 & \lambda a33 \end{bmatrix}$$

The multiplication of two symbols with different indices is called *outer multiplication*, and a simple example is given by

$$aibj = \begin{bmatrix} a1b1 & a1b2 & a1b3 \\ a2b2 & a2b2 & a2b3 \\ a3b1 & a3b2 & a3b3 \end{bmatrix}$$

The previous operations obey usual commutative, associative, and distributive laws, for example:

$$a_i + b_i = b_i + a_i$$
$$a_{ij}\, b_k = b_k\, a_{ij}$$
$$a_i + (b_i + c_i) = (a_i + b_i) + c_i$$
$$a_i\,(b_{jk}\, c_l) = (a_i\, b_{jk})\, c_l$$
$$a_{ij}\,(b_k + c_k) = a_{ij}\, b_k + a_{ij}\, c_k$$

Note that the simple relations $a_i = b_i$ and $a_{ij} = b_{ij}$ imply that $a_1 = b_1$, $a_2 = b_2$, . . . and $a_{11} = b_{11}$, $a_{12} = b_{12}$, . . . However, relations of the form $a_i = b_j$ or $a_{ij} = b_{kl}$ have ambiguous meaning because the distinct indices on each term are not the same, and these types of expressions are to be avoided in this notational scheme. In general, the distinct subscripts on ball individual terms in an equation should match.

It is convenient to adopt the convention that if a subscript appears twice in the same term, then *summation* over that subscript from one to three is implied; for example:

$$a_{ii}=\sum_{i=1}^{3} a_{ii} = a_{11}+a_{22}+a_{33}$$
$$a_{ij}\, b_j= \sum_{j=1}^{3} a_{ij}\, b_j = a_{i1}\, b_1+a_{i2}\, b_2+a_{i3}\, b_3$$

It should be apparent that $a_{ii} = a_{jj} = a_{kk}=$ . . . , and therefore the repeated subscripts or indices are sometimes called dummy subscripts. Unspecified indices that are not repeated are called free or distinct subscripts. The summation convention may be suspended by underlining one of the repeated indices or by writing *no sum*. The use of three or more repeated indices in the same term (e.g., a_{iii} or $a_{iij}\, b_{ij}$) has ambiguous meaning and is to be avoided. On a given symbol, the process of setting two free indices equal is called contraction. For example, a_{ii} is obtained from a_{ij} by contraction on i and j. The operation of outer multiplication of two indexed symbols followed by contraction with respect to one index from each symbol generates an *inner multiplication*; for example, $a_{ij}\, b_{jk}$ is an inner product obtained from the outer product $a_{ij}\, b_{mk}$ by contraction on indices j and m.

A symbol $a_{ij...m...n...k}$ is said to be symmetric with respect to index pair mn if

$$a_{ij...m...n...k} = a_{ij...n...m...k}$$

while it is *antisymmetric* or *skewsymmetric* if

$$a_{ij...m...n...k} = - a_{ij...n...m...k}$$

Note that if $a_{ij...m...n...k}$ is symmetric in mn while $b_{pq...m...n...r}$ is antisymmetric in mn, then the product is zero:

$$a_{ij...m...n...k}\, b_{pq...m...n...r} = 0$$

A useful identity may be written as

$$a_{ij}= \frac{1}{2}\,(a_{ij} + a_{ji}) + \frac{1}{2}\,(a_{ij} - a_{ji}) = a_{(ii)} + a_{[ij]}$$

The first term $a_{(ij)} = 1/2(a_{ij} + a_{ji})$ is symmetric, while the second term $a_{[ij]} =1/2\,(a_{ij} - a_{ji})$ is antisymmetric, and thus an arbitrary symbol a_{ij} can be expressed as the sum of symmetric and antisymmetric pieces. Note that if a_{ij} is symmetric, it has only six independent components. On the other hand, if a_{ij} is antisymmetric, its diagonal terms a_{ii} (no sum on i) must be zero, and it has only three independent components. Note that since $a_{[ij]}$ has only three independent components, it can be relat d to a quantity e with a single index.

1.1.3. Kronecker Delta and Alternating Symbol

A useful special symbol commonly used in index notational schemes is the **Kronecker** delta defined by

$$\delta_{ij} =\begin{cases} 1, if\ i = j\ (\,no\ sun) \\ 0, if\ i{\neq}j \end{cases} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Within usual matrix theory, it is observed that this symbol is simply the unit matrix. Note that the **Kronecker** delta is a symmetric symbol. Particular useful properties of the **Kronecker** delta include the following:

$$\delta_{ij}= \delta_{ji}$$
$$\delta_{ii} =3, \quad \delta_{i\underline{i}}=1$$
$$\delta_{ij}\, a_i = a_j\,, \quad \delta_{ij}\, a_i= a_j$$
$$\delta_{ij}\, a_{jk} = a_{ik}\,, \quad \delta_{jk}\, a_{ik}= a_{ij}$$
$$\delta_{ij}\, a_{ij} = a_{ii}\,, \quad \delta_{ij}\, \delta_{ij} = 3$$

Another useful special symbol is the alternating or permutation symbol defined by

$$\varepsilon_{ijk} = \begin{cases} +1, \text{if ijk is an even permutation of 1,2,3} \\ -1, \text{if ijk is an odd permutation of 1,2,3} \\ \quad 0, \text{otherwise} \end{cases}$$

Consequently, $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$, $\varepsilon_{321} =\varepsilon_{132} = \varepsilon_{213} =-1$, $\varepsilon_{112} = \varepsilon_{131} = \varepsilon_{222} =$ . . . = 0. Therefore, of the 27 possible terms for the alternating symbol, 3 are equal to +1, three are equal to –1, and all others are 0. The alternating symbol is antisymmetric with respect to any pair of its indices.

This particular symbol is useful in evaluating determinants and vector cross products, and the determinant of an array a_{ij} can be written in two equivalent forms:

$$\det\,[a_{ij}]= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \varepsilon_{ijk}\, a_{1i}\, a_{2j} a_{3k} = \varepsilon_{ijk}\, a_{i1}\, a_{j2} a_{k3}$$

where the first index expression represents the row expansion, while the second form is the column expansion. Using the property

$$\varepsilon_{ijk}\, \varepsilon_{pqr} =]= \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix}$$

another form of the determinant of a matrix can be written as

$$\det[a_{ij}] = 1/6\,(\varepsilon_{ijk}\,\varepsilon_{pqr}\,a_{ip}\,a_{jq}\,a_{kr}]$$

1.1.4. Coordinate Transformations

It is convenient and in fact necessary to express elasticity variables and field equations in several different coordinate systems. This situation requires the development of particular transformation rules for scalar, vector, matrix, and higher-order variables. This concept is fundamentally connected with the basic definitions of tensor variables and their related tensor transformation laws. We restrict our discussion to transformations only between Cartesian coordinate systems, and thus consider the two systems shown in Figure 1-1. The two Cartesian frames (x_1, x_2, x_3) and (x'_1, x'_2, x'_3) differ only by orientation, and the unit basis vectors for each frame are $\{e_i\} = \{e_1, e_2, e_3\}$ and $\{e'_i\} = \{e'_1, e'_2, e'_3\}$.

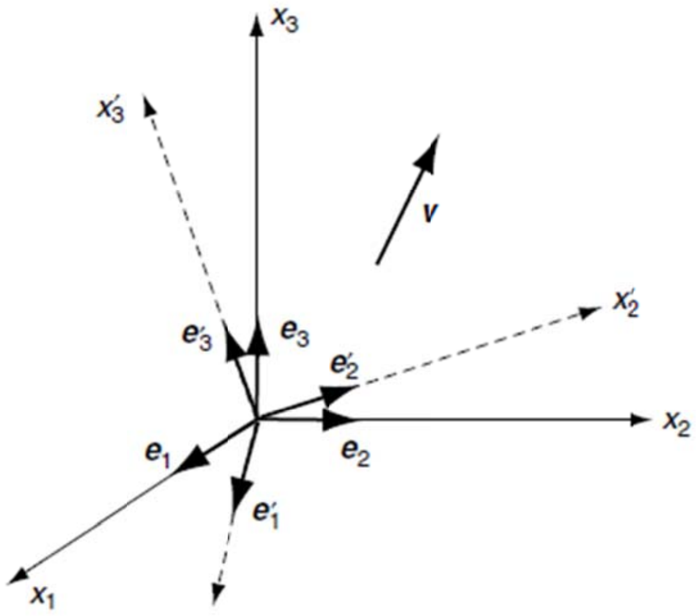

Figure 1-1 Change of Cartesian coordinate frames.

Let Q_{ij} denote the cosine of the angle between the x'_i-axis and the x_j-axis:

$$Q_{ij} = \cos(x'_i, x_j)$$

Using this definition, the basis vectors in the primed coordinate frame can be easily expressed in terms of those in the unprimed frame by the relations

$$e'_1 = Q_{11}e_1 + Q_{12}e_2 + Q_{13}e_3$$
$$e'_2 = Q_{21}e_1 + Q_{22}e_2 + Q_{23}e_3$$
$$e'_3 = Q_{31}e_1 + Q_{32}e_2 + Q_{33}e_3$$

or in index notation

$$e'_i = Q_{ij}e_j$$

Likewise, the opposition transformation can be written using the same format as

$$e_i = Q_{ij}e'_j$$

Now an arbitrary vector v can be written in either of the two coordinate systems as

$$v = v_1e_2 + v_2e_2 + v_3e_3$$
$$= v'_1e'_2 + v'_2e'_2 + v'_3e'_3 = v'_i e'_i$$

Substuting *from* ($e'_i = Q_{ij}e_j$) into $\begin{cases} v = v_1e_1 + v_2e_2 + v_3e_3 \\ = v'_1e'_2 + v'_2e'_2 + v'_3e'_3 = v'_i e'_i \end{cases}$ gives

$$v = v_iQ_{ij}e'_j$$

but from , $v = v'_je'_j$, and so we find that

$$v'_i = Q_{ij}v_j$$

In similar fashion, using the equations gives

$$v_i = Q_{ji}v'_j$$

both last relations constitute the transformation laws for the Cartesian components of a vector under a change of rectangular Cartesian coordinate frame. It should be understood that under such transformations, the vector is unaltered (retaining original length and orientation), and only its components are changed. Consequently, if we know the components of a vector in one frame, relation ($v'_i = Q_{ij}v_j$) and/or relation ($v_i = Q_{ji}v'_j$) can be used to calculate components in any other frame.

The fact that transformations are being made only between orthogonal coordinate systems places some particular restrictions on the transformation or direction cosine matrix Q_{ij}. These can be determined by using ($v'_i = Q_{ij}v_j$) and ($v_i = Q_{ji}v'_j$) together to get

$$v_i = Q_{ji}v'_j = Q_{ji}Q_{jk}v_k$$

From the properties of the **Kronecker** delta, this expression can be written as

$$\delta_{ik}v_k = Q_{ji}Q_{jk}v_k \text{ or } (Q_{ji}Q_{jk} - \delta_{ik})v_k = 0$$

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and since this relation is true for all vectors v_k, the expression in parentheses must be zero, giving the result

$$Q_{ji}\,Q_{jk} = \delta_{ik}$$

In similar fashion, relations ($v'_i = Q_{ij}v_j$) and ($v_i = Q_{ji}\,v'_j$) can be used to eliminate v_i (instead of v'_i) to get

$$v_i = Q_{ji}\,v'_j = Q_{ji}\,Q_{jk}\,v_k$$

From the properties of the **Kronecker** delta, this expression can be written as

$$\delta_{ik}\,v_k = Q_{ji}\,Q_{jk}\,v_k \text{ or } (Q_{ji}\,Q_{jk} - \delta_{ik})\,v_k = 0$$

and since this relation is true for all vectors v_k, the expression in parentheses must be zero, giving the result

$$Q_{ji}\,Q_{jk} = \delta_{ik}$$

In similar fashion, relations ($v'_i = Q_{ij}v_j$) and ($v_i = Q_{ji}\,v'_j$) can be used to eliminate vi (instead of v'i) to get

$$Q_{ij}\,Q_{kj} = \delta_{ik}$$

Relations ($Q_{ji}\,Q_{jk} = \delta_{ik}$) and ($Q_{ij}\,Q_{kj} = \delta_{ik}$) comprise the orthogonality conditions that Q_{ij} must satisfy.bTaking the determinant of either relation gives another related result:

$$\det[Q_{ij}] = \pm 1$$

Matrices that satisfy these relations are called orthogonal, and the transformations given by ($v'_i = Q_{ij}v_j$) and ($v_i = Q_{ji}\,v'_j$) are therefore referred to as orthogonal transformations.

1.1.5. Cartesian Tensors

Scalars, vectors, matrices, and higher-order quantities can be represented by a general index notational scheme. Using this approach, all quantities may then be referred to as tensors of different orders. The previously presented transformation properties of a vector can be used to establish the general transformation properties of these tensors. Restricting the transformations to those only between Cartesian coordinate systems, the general set of transformation relations for various orders can be written as

$$\begin{aligned}
a' &= a, \text{ zero order (scalar)}\\
a'_i &= Q_{ip}\,a_p, \text{ first order (vector)}\\
a'_{ij} &= Q_{ip}\,Q_{jq}\,a_{pq}, \text{ second order (matrix)} \qquad (\mathbf{1.1})\\
a'_{ijk} &= Q_{ip}\,Q_{jq}\,Q_{kr}\,a_{pqr}, \text{ third order}\\
a'_{ijkl} &= Q_{ip}\,Q_{jq}\,Q_{kr}\,Q_{ls}\,a_{pqrs}, \text{ fourth order}\\
&\;\;\vdots\\
a'_{ijk\ldots m} &= Q_{ip}\,Q_{jq}\,Q_{kr}\,_\,_\,_\,Q_{mt}\,a_{pqr\ldots t} \text{ general order}
\end{aligned}$$

Note that, according to these definitions, a scalar is a zero-order tensor, a vector is a tensor of order one, and a matrix is a tensor of order two. Relations (**1.1**) then specify the transformation rules for the components of Cartesian tensors of any order under the rotation Q_{ij}. This transformation theory proves to be very valuable in determining the displacement, stress, and strain in different coordinate directions. Some tensors are of a special form in which their components remain the same under all transformations, and these are referred to *as isotropic tensors*. It can be easily verified that the **Kronecker** delta δ_{ij} has such a property and is therefore a second-order *isotropic tensor*. The alternating symbol ε_{ijk} is found to be the third-order isotropic form. The fourth order case can be expressed in terms of products of **Kronecker** deltas.

The distinction between the components and the tensor should be understood. Recall that a vector v can be expressed as

$$\begin{aligned}
v &= v_1e_1 + v_2e_2 + v_3e_3 = v_ie_i\\
&= v'_1e'_1 + v'_2e'_2 + v'_3e'_3 = v'_ie'_i \qquad (\mathbf{1.2})
\end{aligned}$$

In a similar fashion, a second-order tensor A can be written

$$\begin{aligned}
A &= A_{11}e_1e_1 + A_{12}e_1e_2 + A_{13}e_1e_3 + A_{21}e_2e_1 + A_{22}e_2e_2 + A_{23}e_2e_3 + A_{31}e_3e_1 + A_{32}e_3e_2 + A_{33}e_3e_3\\
&= A_{ij}\,e_i\,e_j = A'_{ij}\,e'_i\,e'_j \qquad (\mathbf{1.3})
\end{aligned}$$

and similar schemes can be used to represent tensors of higher order. The representation used in equation (**3**) is commonly called dyadic notation, and some authors write the dyadic products $e_i\,e_j$ using a tensor product notation $e_i \otimes e_j$.

Relations (**1.2**) and (**1.3**) indicate that any tensor can be expressed in terms of components in any coordinate system, and it is only the components that change under coordinate transformation. For example, the state of stress at a point in an elastic solid depends on the problem geometry and applied loadings. As is shown later, these stress components are those of a second-order tensor and therefore obey transformation law (**1.1**). Although the components of the stress tensor change with the choice of coordinates, the stress tensor (representing the state of stress) does not.

An important property of a tensor is that if we know its components in one coordinate system, we can find them in any other coordinate frame by using the appropriate transformation law. Because the components of Cartesian tensors are representable by indexed symbols, the operations of equality, addition, subtraction, multiplication, and so forth are defined in a manner consistent with the indicial notation procedures previously discussed. The terminology tensor without the adjective *Cartesian* usually refers to a more general scheme in which the coordinates are not necessarily rectangular *Cartesian* and the transformations between coordinates are not always orthogonal. Such general tensor theory is not discussed or used in this text.