Exercise Sheet 2

Exercise 01

1) Truth Tables

$\mathbf{F}_1 : p \wedge (\neg q \Rightarrow (q \Rightarrow p))$

p	q	$q \Rightarrow p$	$\neg q$	$\neg q \Rightarrow (q \Rightarrow p)$	F_1
0	0	1	1	1	0
0	1	0	0	1	0
1	0	1	1	1	1
1	1	1	0	1	1

$\mathbf{F}_2 : (p \vee q) \Leftrightarrow (\neg p \Rightarrow \neg q)$

p	q	$p \vee q$	$\neg p$	$\neg q$	$\neg p \Rightarrow \neg q$	F_2
0	0	0	1	1	1	0
0	1	1	1	0	0	0
1	0	1	0	1	1	1
1	1	1	0	0	1	1

$$\mathrm{Mod}(F_2) = \{(I(p) = 1, I(q) = 0), (I(p) = 1, I(q) = 1)\}$$

3) Truth values

- $I(p) = 0, I(q) = 1$: $F_1 = 0,\ F_2 = 0$
- $I(p) = 0$: $F_1 = 0,\ F_2 = 0$
- $I(q) = 1$: depends on p
- No information: depends on p

Exercise 02

Definition: A propositional formula F is said to be a **tautology (valid)** if and only if its truth value is always **true**.

1. Using equivalent transformations.

   $\mathbf{F}_1 : \neg(p \Rightarrow q) \Rightarrow \neg q$

$$p \Rightarrow q \equiv \neg p \vee q$$

$$\neg(p \Rightarrow q) \equiv p \wedge \neg q$$

$$(p \wedge \neg q) \Rightarrow \neg q \equiv \neg(p \wedge \neg q) \vee \neg q \equiv (\neg p \vee q) \vee \neg q \equiv T$$

$$\Rightarrow \text{Tautology}$$

$\mathbf{F}_2 : (p \Rightarrow q) \vee (q \Rightarrow p)$

$$p \Rightarrow q \equiv \neg p \vee q$$
$$q \Rightarrow p \equiv \neg q \vee p$$
$$(p \Rightarrow q) \vee (q \Rightarrow p) \equiv (\neg p \vee q) \vee (\neg q \vee p) \equiv (\neg p \vee p) \vee (\neg q \vee q) \equiv T \vee T \equiv T$$
$$\Rightarrow \text{Tautology}$$

2. Using truth table.

$\mathbf{F}_1 : \neg(p \Rightarrow q) \Rightarrow \neg q$

p	q	$p \Rightarrow q$	$\neg(p \Rightarrow q)$	$\neg q$	F_1
0	0	1	0	1	1
0	1	1	0	0	1
1	0	0	1	1	1
1	1	1	0	0	1

$\mathbf{F}_1$ is true in all interpretations, then F_1 is a Tautology.

$\mathbf{F}_2 : (p \Rightarrow q) \vee (q \Rightarrow p)$

p	q	$p \Rightarrow q$	$q \Rightarrow p$	F_2
0	0	1	1	1
0	1	1	0	1
1	0	0	1	1
1	1	1	1	1

$\mathbf{F}_2$ is true in all interpretations, then F_2 is a Tautology.

Exercise 03

1) $F_1 = p \vee (p \wedge q)$ and $F_2 = p$

p	q	$p \wedge q$	F_1	F_2
0	0	0	0	0
0	1	0	0	0
1	0	0	1	1
1	1	1	1	1

$$F_1 \equiv F_2$$

2) $F_3 = p \wedge (q \vee r)$ and $F_4 = (p \wedge q) \vee (p \wedge r)$

p	q	r	$q \vee r$	F_3	$p \wedge q$	$p \wedge r$	F_4
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	1	0	1	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

$$F_3 \equiv F_4$$

3) $F_5 = (p \Rightarrow q) \Rightarrow r$ **and** $F_6 = p \Rightarrow (q \Rightarrow r)$

p	q	r	$p \Rightarrow q$	F_5	$q \Rightarrow r$	F_6
0	0	0	1	0	1	1
0	0	1	1	1	1	1
0	1	0	1	0	0	1
0	1	1	1	1	1	1
1	0	0	0	1	1	1
1	0	1	0	1	1	1
1	1	0	1	0	0	0
1	1	1	1	1	1	1

Since the truth values differ in the first and third rows:

$$F_5 \not\equiv F_6$$

Exercise 04

I. Convert to CNF & DNF using truth table

1) $F_1 = (p \wedge q) \Leftrightarrow (\neg p \Rightarrow q)$

Truth Table

p	q	F_1
0	0	1
0	1	0
1	0	0
1	1	1

DNF
True rows: $(0,0),(1,1)$

$$\boxed{(\neg p \wedge \neg q) \vee (p \wedge q)}$$

CNF
False rows: $(0,1),(1,0)$

$$\boxed{(p \vee \neg q) \wedge (\neg p \vee q)}$$

2) $F_2 = (\neg p \Rightarrow q) \Rightarrow (q \Rightarrow \neg r)$

Truth Table

p	q	r	F_2
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0

DNF
True rows:
$(0,0,0), (0,0,1), (0,1,0), (1,0,0), (1,0,1), (1,1,0)$

$$\boxed{(\neg p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge q \wedge \neg r)}$$

CNF
False rows: $(0,1,1), (1,1,1)$

$$\boxed{(p \vee \neg q \vee \neg r) \wedge (\neg p \vee \neg q \vee \neg r)}$$

3) $F_3 = (\neg p \vee q) \vee (p \Rightarrow r)$

Truth Table

p	q	r	F_3
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

DNF
True rows:
$(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,1), (1,1,0), (1,1,1)$

$$\boxed{(\neg p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge q \wedge r)}$$

CNF
False row: $(1,0,0)$

$$\boxed{(\neg p \vee q \vee r)}$$

II. Convert to CNF using equivalences

$F_4 = \neg(\neg p \vee q) \wedge (\neg r \Rightarrow \neg s)$

$$\neg(\neg p \vee q) \equiv p \wedge \neg q$$

$$\neg r \Rightarrow \neg s \equiv r \vee \neg s$$

$$\boxed{F_4 = (p) \wedge (\neg q) \wedge (r \vee \neg s)}$$

$F_5 = (p \wedge q) \vee (\neg p \vee q)$

$$(p \wedge q) \vee q \equiv q$$

$$\boxed{F_5 = (\neg p \vee q)}$$

$F_6 = (p \wedge \neg q) \Rightarrow \neg r$

$$A \Rightarrow B \equiv \neg A \vee B$$

$$F_6 = \neg(p \wedge \neg q) \vee \neg r$$

$$\neg(p \wedge \neg q) \equiv \neg p \vee q$$

$$\boxed{F_6 = (\neg p \vee q \vee \neg r)}$$

Exercise 05

1) $F_1 = (p \Leftrightarrow q) \Rightarrow ((p \Rightarrow q) \vee (q \Rightarrow p))$

p	q	$p \Leftrightarrow q$	$p \Rightarrow q$	$q \Rightarrow p$	F_1
0	0	1	1	1	1
0	1	0	1	0	1
1	0	0	0	1	1
1	1	1	1	1	1

$\boxed{\text{Tautology}}$

2) $F_2 = (p \Rightarrow q) \Rightarrow (q \vee \neg p)$

p	q	$p \Rightarrow q$	$q \vee \neg p$	F_2
0	0	1	1	1
0	1	1	1	1
1	0	0	0	1
1	1	1	1	1

$\boxed{\text{Tautology}}$

3) $F_3 = p \wedge \neg(q \vee p)$

p	q	$q \vee p$	$\neg(q \vee p)$	F_3
0	0	0	1	0
0	1	1	0	0
1	0	1	0	0
1	1	1	0	0

$\boxed{\text{Unsatisfiable}}$

4) $F_4 = (p \wedge q) \wedge \neg p$

p	q	$p \wedge q$	$\neg p$	F_4
0	0	0	1	0
0	1	0	1	0
1	0	0	0	0
1	1	1	0	0

$\boxed{\text{Unsatisfiable}}$

5) $F_5 = (p \Rightarrow \neg q) \wedge q$

p	q	$\neg q$	$p \Rightarrow \neg q$	F_5
0	0	1	1	0
0	1	0	1	1
1	0	1	1	0
1	1	0	0	0

$\boxed{\text{Satisfiable but Not Valid}}$

Final Classification

- F_1: Tautology (Valid)
- F_2: Tautology (Valid)
- F_3: Unsatisfiable (Contradiction)
- F_4: Unsatisfiable (Contradiction)
- F_5: Satisfiable but not Valid