Mathematical Logic

Lecture 8: Predicate Logic — Semantics

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Level: First Year LMD — Licence Computer Science

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Idea

Goal: When is a formula TRUE or FALSE?

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Well-formed formula + Interpretation + Values = Truth Value

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- Constants:

$$I(a) \in D$$

Example of Interpretation

$$F = P(f(x,y),y)$$

Interpretation 1:

- $D = \mathbb{Z}$
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- $P(x,y) : x > y$

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Interpretation 2:

- $D = \mathbb{R}$
- $f(x, y) = \frac{x^2 - y}{2}$
- $P(x, y) : x = y$

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- **False formula:** A formula A is true for an interpretation I if $v(A) = 0$ (False) in I.

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- $v(x) = 3,\ v(y) = 2 \Rightarrow$ the truth value of F is: False

Quantifiers

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- To prove that a formula as $\forall x P(x)$ is false, it is sufficient to verify it with a counterexample.

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Universal:

$$x = 2 \Rightarrow \forall x P(x) \text{ is false}$$

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- The formula F is satisfiable.

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$$F = \exists x(P(x, y) \wedge \neg P(x, y))$$

- F is Always false.
- F is an unsatisfiable formula.

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$$F = \forall x \forall y((x \neq y) \vee P(x, y) \vee \neg P(y, x))$$

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There are two possible cases, whatever the interpretation I considered:

- If $x = y$: $P(x,y) \vee \neg P(y,x) = P(x,x) \vee \neg P(x,x)$ is always true.
- If $x \neq y$: is true because the first predicate is true.

So F is valid ($\models F$).

Prenex Normal Form

Definition: A formula F is said to be in prenex form iff it is of the form:

$$Q_1x_1 \ldots Q_nx_nA'$$

- Q_i : a quantifier $\exists$ or $\forall$.
- A' : formula does not contain quantifiers.

Theorem: For every formula A there exists a formula A' in prenex form such as: $A \equiv A'$.

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$$\forall x \forall z(\exists y P(x,y) \wedge \exists y Q(z,y)) \equiv \forall x \forall z(\exists y P(x,y) \wedge \exists w Q(z,w))$$

Eliminate implications and equivalences

As in propositional logic:

$$P \Rightarrow Q \equiv \neg P \vee Q$$

$$P \Leftrightarrow Q \equiv (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

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- Prenex:

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$$\forall y \exists x A(x,y) \rightarrow \forall y A(f(y),y)$$

$$\forall y \forall z \exists x A(x,y,z) \rightarrow \forall y A(f(y,z),y,z)$$

Conjunctive Normal Form

- Convert to SNF

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- Distribute $\vee$ over $\wedge$

$$(P \wedge Q) \vee R \equiv (P \vee R) \wedge (Q \vee R)$$

CNF Example

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Exercise

Convert to CNF:

$$\forall x(\forall y(P(y) \Rightarrow Q(x,y))) \Rightarrow (\exists y Q(y,x))$$

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5. CNF:

$$(P(f(x)) \vee Q(g(x), x)) \wedge (\neg Q(x, f(x)) \vee Q(g(x), x))$$