# Mathematical Logic

## Lecture 7: Predicate Logic — Syntax

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Level: First Year LMD — Licence Computer Science

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# Motivation

# Why Predicate Logic?

**Example:**

- Ibrahim is a student
- Zahra is a student
- Imad is a student

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Predicate = generalization of propositions

# Proposition vs Predicate

## Proposition

Has a fixed truth value

$$5 > 3 \Rightarrow \text{True}$$

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Predicate becomes a proposition when $x$ is instantiated

**Example:**

- $5 > 3$ It is a statement, we can model it as a proposition, so its truth value is true.
- $2 > 3$ It is a statement, we can model it as a proposition, so its truth value is false.
- $x > 3$ This is a statement, we cannot model it as a proposition. The truth value depends on the value of $x$.

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**Example:**

Consider the following sentences:

- Person $x$ is father of $y$.
- Person $z$ is father of $x$.
- So, person $z$ is grandfather of $y$.

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Consider the following sentences:

- Person $x$ is father of $y$.
- Person $z$ is father of $x$.
- So, person $z$ is grandfather of $y$.

It is very difficult to translate these sentences into propositional logic.

# Basic Concepts

# Domain (Universe)

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**Examples:**

- $D = \mathbb{N}$
- $D =$ set of students
- $D =$ set of people

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- Predicates: $P(x), Q(x, y)$

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- Functions: $f(x), g(x, y)$
- Predicates: $P(x), Q(x, y)$
- Quantifiers: $\forall, \exists$

# Terms

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**Examples:**

$$x,\ A,\ f(x),\ g(y, f(x))$$

# Predicates

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$$D^n \to \{0, 1\}$$

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**Examples:**

- Student(x)

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Number of arguments = arity

# Quantifiers

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**Important**

Use $\Rightarrow$ with $\forall$

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" There are students who are intelligent" is modeled by:

$$\exists x(Student(x) \wedge Intelligent(x))$$

" Some $A(x)$ are $B(x)$" is modeled by:

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**Important**

Use $\wedge$ with $\exists$

# Important Note

If domain is finite:

$$\forall x P(x) \equiv P(n_1) \land \cdots \land P(n_k)$$

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$$\exists x P(x) \equiv P(n_1) \lor \cdots \lor P(n_k)$$

# Important Note

If domain is finite:

$$\forall x P(x) \equiv P(n_1) \wedge \cdots \wedge P(n_k)$$

$$\exists x P(x) \equiv P(n_1) \vee \cdots \vee P(n_k)$$

Only valid for finite domains!

# Order of Quantifiers

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# Negation of Quantifiers

$$\neg\forall x P(x) \equiv \exists x \neg P(x)$$

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# Formulas

**Definition**

If $P$ is a predicate and $t_i$ are terms:

$$P(t_1, \ldots, t_n)$$

is an atomic formula.

# Atomic Formulas

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**Examples:**

$$Student(Amine), \quad Greater(x, y)$$

# Well-Formed Formulas (WFF)

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2. If $A, B$ are WFF:

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   are well-formed formulas.
3. If $A$ is a formula and $x$ a variable:

$$\forall x(A), \quad \exists x(A)$$

   are well-formed formulas.

# Free and Bound Variables

**Definition**

- Bound variable: inside quantifier scope

# Free vs Bound

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**Example:**

$$\forall x P(x) \wedge Q(x)$$

$x$ is bound in $P(x)$ but free in $Q(x)$

# Open vs Closed Formulas

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**Examples:**

$$\forall x P(x) \Rightarrow \text{closed}$$

$$P(x) \Rightarrow \text{open}$$