Mathematical Logic

Lecture 3: Semantics - Part 2

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Normal Forms

Why Normal Forms?

We need systematic methods to:

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We need systematic methods to:

- Test satisfiability

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Solution:

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Normal forms are built from literals.

Definition

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Example

$$(p_1 \vee p_2 \vee \neg p_3) \wedge (\neg p_1 \vee p_4) \wedge (\neg p_1 \vee p_3 \vee \neg p_4)$$

Definition

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Example

$$(\neg p_1 \wedge p_2 \wedge p_4) \vee (p_2 \wedge \neg p_3)$$

Properties of Normal Forms

- Every formula is logically equivalent to its Normal Forms:
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- If a formula F is unsatisfiable (an antilogy), then $\mathrm{CNF}(F) \equiv 0$.
- If a formula F is valid (a tautology), then $\mathrm{DNF}(F) \equiv 1$.

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- Use the minterms / maxterms method.

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- Get rid of double negations.
- Distributivity: Convert $p \vee (q \wedge r)$ to $(p \vee q) \wedge (p \vee r)$ for CNF or inverse for DNF.

Example

Convert the following formulas into CNF: $F : ((p \vee q) \Rightarrow r) \vee s$

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Example

Convert the following formulas into CNF: $F : ((p \lor q) \Rightarrow r) \lor s$

$$F \equiv (\neg(p \lor q) \lor r) \lor s \quad \text{Material implication}$$

$$F \equiv ((\neg p \land \neg q) \lor r) \lor s \quad \text{De Morgan's Law}$$

$$F \equiv ((\neg p \lor r) \land (\neg q \lor r)) \lor s \quad \text{Distributivity (OR over AND)}$$

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Final CNF

$$\boxed{(\neg p \vee r \vee s) \wedge (\neg q \vee r \vee s)} \quad \text{Associativity of OR}$$

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- DNF: Formula as a disjunction of minterms.
- CNF: Formula as conjunction of maxterms.

Example

p	q	Formula	Minterm	Maxterm
0	0	1	$\neg p \wedge \neg q$	
0	1	1	$\neg p \wedge q$	
1	0	0		$\neg(p \wedge \neg q) \equiv \neg p \vee q$
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DNF : $(\neg p \wedge \neg q) \vee (\neg p \wedge q)$.

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DNF : $(\neg p \wedge \neg q) \vee (\neg p \wedge q)$.
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0	1	0	1	$\neg p \wedge q$	
1	0	0	1	$p \wedge \neg q$	
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DNF : $(\neg p \wedge \neg q) \vee (\neg p \wedge q) \vee (p \wedge \neg q)$.

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1	0	0	1	$p \wedge \neg q$	
1	1	1	0		$\neg p \vee \neg q$

DNF : $(\neg p \wedge \neg q) \vee (\neg p \wedge q) \vee (p \wedge \neg q)$.
CNF : $(\neg p) \vee (\neg q)$.

Satisfiability of a Set of Formulas

Definition

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- A set Σ is said to be satisfiable if and only if given the truth table of all formulas, there exists at least one row where all these formulas $P_1, P_2, \ldots, P_n$ are true simultaneously.

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- A set Σ is said to be satisfiable if and only if given the truth table of all formulas, there exists at least one row where all these formulas $P_1, P_2, \ldots, P_n$ are true simultaneously.

$$\Sigma \text{ is satisfiable iff : } \exists I \text{ such that } I(P_i) = 1 \quad \forall P_i \in \Sigma$$

Example

Consider:

$$\Sigma = \{p \Rightarrow q,\ p \wedge q,\ p \vee q\}$$

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p	q	$p \Rightarrow q$	$p \wedge q$	$p \vee q$
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0	1	1	0	1
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0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

All formulas are true when:

$$I(p) = 1,\ I(q) = 1$$

$\Rightarrow$ *Satisfiable*

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Truth table:

p	q	$p \Rightarrow q$	$q \Rightarrow \neg p$
0	0	1	1
0	**1**	**1**	**1**
1	0	0	1
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Truth table:

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0	0	1	1
0	**1**	**1**	**1**
1	0	0	1
1	1	1	0

All formulas are true when:

$$I(p) = 0,\ I(q) = 1$$

$\Rightarrow$ *Satisfiable*

Logical Consequence

Definition

- Q is said to be a semantic consequence of P if every model of P is a model of Q.

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- We note: $P \models Q$.

Example

Show that:

$$p \wedge q \models p \vee q$$

Example

Show that:

$$p \land q \models p \lor q$$

Truth table:

p	q	$p \land q$	$p \lor q$
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	1

Example

Show that:

$$p \land q \models p \lor q$$

Truth table:

p	q	$p \land q$	$p \lor q$
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	1

Whenever $p \land q = 1$, we also have $p \lor q = 1$.

$\Rightarrow$ Valid consequence

Property

Let $P \models Q$ be if and only if $P \Rightarrow Q$ is valid.

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Example

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Truth table:

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \Rightarrow (p \vee q)$
0	0	0	0	1
0	1	0	1	1
1	0	0	1	1
1	1	1	1	1

Definition

- We say that Q is a semantic consequence of (or of the set $\Sigma = \{P_1, P_2, \ldots, P_n\}$) if in the truth table of Q is true whenever $P_1, P_2, \ldots, P_n$ are true.

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- We note: $\{P_1, P_2, \ldots, P_n\} \models Q$ or $\Sigma \models Q$.

To check the consequence, we must check the truth value of Q is true on all lines where the formulas $P_1, P_2, \ldots, P_n$ are simultaneously true.

Example

Show that:

$$\{(p \Rightarrow \neg q), q\} \models \neg p$$

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Truth table:

p	q	$p \Rightarrow \neg q$	$\neg p$
0	0	1	1
0	**1**	**1**	**1**
1	0	1	0
1	1	0	0

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Truth table:

p	q	$p \Rightarrow \neg q$	$\neg p$
0	0	1	1
0	**1**	**1**	**1**
1	0	1	0
1	1	0	0

Whenever $p \Rightarrow \neg q$ and q are simultaneously true, we also have $\neg p$ is true.
$\Rightarrow$ Valid consequence

Properties

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1. $$\Sigma \models Q \text{ if and only if } (P_1 \wedge \cdots \wedge P_n \Rightarrow Q) \text{ is a tautology}$$

Properties

Let $\Sigma = \{P_1, P_2, \dots, P_n\}$ be a set of formulas.

1. $$\Sigma \models Q \text{ if and only if } (P_1 \wedge \cdots \wedge P_n \Rightarrow Q) \text{ is a tautology}$$

2. $$\Sigma \models Q \text{ if and only if } \Sigma \cup \{\neg Q\} \text{ is a is unsatisfiable}$$

Example

Show that:

$$\{(p \Rightarrow \neg q), q\} \models \neg p$$

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Truth table:

p	q	$p \Rightarrow \neg q$	$\neg p$	$(p \Rightarrow \neg q) \wedge q$	$((p \Rightarrow \neg q) \wedge q) \Rightarrow \neg p$
0	0	1	1	0	1
0	1	1	1	1	1
1	0	1	0	0	1
1	1	0	0	1	1

Example

Show that:

$$\{(p \Rightarrow \neg q), q\} \models \neg p$$

Truth table:

p	q	$p \Rightarrow \neg q$	$\neg p$	$(p \Rightarrow \neg q) \wedge q$	$((p \Rightarrow \neg q) \wedge q) \Rightarrow \neg p$
0	0	1	1	0	1
0	1	1	1	1	1
1	0	1	0	0	1
1	1	0	0	1	1

$((p \Rightarrow \neg q) \wedge q) \Rightarrow \neg p$ is a tautology

$\Rightarrow$ Valid consequence

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Truth table:

p	q	$p \Rightarrow \neg q$
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Show that:

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Truth table:

p	q	$p \Rightarrow \neg q$
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0	1	1
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1	1	0

The set $\Sigma \cup \{\neg Q\} = \{(p \Rightarrow \neg q), q, p\}$ is unsatisfiable

$\Rightarrow$ Valid consequence

Exercise: Modus Ponens

Show that:

$$\{p \Rightarrow q,\ p\} \models q$$