MOHAMED BOUDIAF UNIVERSITY OF MSILA

Faculty of Mathematics and Computer Science **Computer Science Department**

Module : Analysis 2 **University year :**2025/2026

Series N°4 : Functions of several variables

Exercise 1 :

Give the domain of definition of the following functions then represent it geometrically.

$$f(x,y) = \frac{1}{x} - \frac{1}{y+1}, \quad g(x,y) = \sqrt{1-xy}, \quad h(x,y) = \ln(xy),$$

$$e(x,y) = \ln[(x^2+y^2-4)(9-x^2-y^2)]$$

Exercise 2 :

Calculate the following limits if there exists

$$\lim_{(x,y)\to(0,5)} \frac{1}{x}\sin(xy), \qquad \lim_{(x,y)\to(0,0)} \frac{1+x^2+y^2}{y}\sin(y), \qquad \lim_{(x,y)\to(\infty,2)} \left(1+\frac{y}{x}\right).$$

Exercise 3 :

Verify if the following functions are continuous at $(0,0)$

$$(x,y) \mapsto f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

$$(x,y) \mapsto g(x,y) = \begin{cases} (x^2+y^2)\sin\dfrac{1}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

$$(x,y) \mapsto h(x,y) = \begin{cases} \dfrac{(x+y)^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

Exercise 4 :

Let f function defined by

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

Calculate

1. $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$.
2. $\dfrac{\partial f}{\partial x}(x,y)$ and $\dfrac{\partial f}{\partial y}(x,y)$ at $(x,y) \neq (0,0)$.
3. $\dfrac{\partial^2 f}{\partial x \partial y}(0,0)$ and $\dfrac{\partial^2 f}{\partial y \partial x}(0,0)$.
4. What do you conclude ?

Dr : M. Touahria

Solution

Exercise 1

We determine the domain of definition of each function and describe it geometrically.

1. **Function** $f(x,y) = \dfrac{1}{x} - \dfrac{1}{y+1}$

   The function is defined when the denominators are nonzero :

   $$x \neq 0 \quad \text{and} \quad y+1 \neq 0 \iff y \neq -1.$$

   Hence, the domain is

   $$D_f = \{(x,y) \in \mathbb{R}^2 \ : \ x \neq 0, \ y \neq -1\}.$$

   Geometric representation : the plane $\mathbb{R}^2$ without the vertical line $x = 0$ and the horizontal line $y = -1$.

2. **Function** $g(x,y) = \sqrt{1 - xy}$

   The square root requires

   $$1 - xy \geq 0 \iff xy \leq 1.$$

   Thus,

   $$D_g = \{(x,y) \in \mathbb{R}^2 \ : \ xy \leq 1\}.$$

   Geometric representation : the region of the plane defined by $xy \leq 1$, i.e., all points on and below the hyperbola $xy = 1 \implies y = \dfrac{1}{x}$.

3. *Function* $h(x,y) = \ln(xy)$

   The logarithm is defined only for strictly positive arguments :

   $$xy > 0.$$

   Hence,

   $$D_h = \{(x,y) \in \mathbb{R}^2 \ : \ xy > 0\}.$$

   Geometric representation : the union of the first and third quadrants (excluding the axes).

4. **Function** $e(x,y) = \ln\left[(x^2 + y^2 - 4)(9 - x^2 - y^2)\right]$

   We require

   $$(x^2 + y^2 - 4)(9 - x^2 - y^2) > 0.$$

   This product is positive when both factors have the same sign :

   $$\left\{ x^2 + y^2 - 4 > 0 \quad \text{and} \quad 9 - x^2 - y^2 > 0 \right.$$

   which gives

   $$4 < x^2 + y^2 < 9.$$

   Thus,

   $$D_e = \{(x,y) \in \mathbb{R}^2 \ : \ 4 < x^2 + y^2 < 9\}.$$

   Geometric representation : the open annulus centered at the origin, between the circles of radius 2 and 3.

   Remark 1 *If*

   $$\left\{ x^2 + y^2 - 4 < 0 \quad \textit{and} \quad 9 - x^2 - y^2 < 0 \right.$$

   $$D_e = \emptyset$$

Dr : M. Touahria

Exercise. 02

We compute each limit separately.

1. $\lim_{(x,y)\to(0,5)} \frac{1}{x}\sin(xy)$

Let $(x,y)\to(0,5)$. Then $xy \to 0$, so we use the standard limit :

$$\lim_{t\to 0} \frac{\sin t}{t} = 1.$$

Write :

$$\frac{1}{x}\sin(xy) = y \cdot \frac{\sin(xy)}{xy}.$$

Thus,

$$\lim_{(x,y)\to(0,5)} \frac{1}{x}\sin(xy) = \lim_{(x,y)\to(0,5)} y \cdot \frac{\sin(xy)}{xy} = 5\cdot 1 = 5.$$

2. $\lim_{(x,y)\to(0,0)} \frac{1+x^2+y^2}{y}\sin(y)$

Rewrite :

$$\frac{1+x^2+y^2}{y}\sin(y) = (1+x^2+y^2)\cdot\frac{\sin(y)}{y}.$$

Using :

$$\lim_{y\to 0}\frac{\sin y}{y} = 1,$$

and $1+x^2+y^2 \to 1$, we obtain :

$$\lim_{(x,y)\to(0,0)} \frac{1+x^2+y^2}{y}\sin(y) = 1\cdot 1 = 1.$$

3. $\lim_{(x,y)\to(\infty,2)} \left(1+\frac{y}{x}\right)$

As $x\to\infty$ and $y\to 2$, we have :

$$\frac{y}{x}\to 0.$$

Therefore,

$$\lim_{(x,y)\to(\infty,2)} \left(1+\frac{y}{x}\right) = 1+0 = 1.$$

1. **Continuity of g at $(0,0)$** We compute the limit :

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right).$$

Using the inequality $|\sin t|\le 1$, we obtain

$$\left|(x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right)\right| \le x^2+y^2.$$

Since $x^2+y^2\to 0$ as $(x,y)\to(0,0)$, by the squeeze theorem,

$$\lim_{(x,y)\to(0,0)} g(x,y) = 0.$$

Or by polar coordinates $x = r\cos\theta, \quad y = r\sin\theta.$

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right) = \lim_{r\to 0} r^2 \sin\frac{1}{r^2} = 0. \quad \text{bounded times zero}$$

Dr : M. Touahria

2. **Continuity of h at** $(0,0)$

   We examine the limit :
$$\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2}.$$
   We use polar coordinates :
$$x = r\cos\theta, \quad y = r\sin\theta.$$
   Then
$$(x+y)^2 = r^2(\cos\theta+\sin\theta)^2, \quad x^2+y^2 = r^2.$$
   Thus,
$$h(x,y) = \frac{r^2(\cos\theta+\sin\theta)^2}{r^2} = (\cos\theta+\sin\theta)^2.$$
   This expression depends on θ, hence it has no unique limit as $r \to 0$.
   For example :
$$\theta = 0 \Rightarrow h \to 1, \qquad \theta = \frac{\pi}{4} \Rightarrow h \to 2.$$
   Since the limit depends on the path, the limit does not exist. Therefore, h is not continuous at $(0,0)$.

Exercise 04

1. The first derivatives at $(0,0)$: we use the definition
$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{y-0} = 0$$
$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y-0} = 0$$
2. Calculate the first derivatives at any point $(x,y) \neq (0,0)$:
$$\frac{\partial f}{\partial x}(x,y) = \frac{y^3(y^2-x^2)}{(x^2+y^2)^2}$$
$$\frac{\partial f}{\partial x}(x,y) = \frac{3x^3y^2+xy^4}{(x^2+y^2)^2}$$
3. The second order derivatives at $(0,0)$: we use the definition
$$\frac{\partial^2 f}{\partial x\partial y}(0,0) = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y})(0,0) = \lim_{x\to 0} \frac{\frac{\partial f}{\partial y}(x,0) - \frac{\partial f}{\partial y}(0,0)}{x-0} = 0,$$
$$\frac{\partial^2 f}{\partial y\partial x}(0,0) = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x})(0,0) = \lim_{x\to 0} \frac{\frac{\partial f}{\partial x}(x,0) - \frac{\partial f}{\partial x}(0,0)}{x-0} = 1,$$
4. Since $\frac{\partial^2 f}{\partial y\partial x}(0,0) \neq \frac{\partial^2 f}{\partial x\partial y}(0,0)$, we conclude that the second order derivatives are not continuous at $(0,0)$.