MOHAMED BOUDIAF UNIVERSITY OF MSILA

Faculty of Mathematics and Computer Science **Computer Science Department**

Module : Analysis 2 **University year :**2026/2027

Series N°3 : Differential equations (*) for student

Exercise 01 : Give the type of each **first-order** differential equations (the Solution is not required)

1. $xy' = (x-1)y, \qquad (1+y^2)dy = xdx, \qquad 2xyy' - y^2 + x = 0.$
2. $y' = \dfrac{x-y}{x+y}, \qquad y' - \dfrac{y}{1-x^2} = 1+x.$

Exercise 02 : Solve the following *separable* differential equations :

- $(1+e^x)yy' = e^x$, ◦ $\tan(x)\sin^2(y)dx + \cos^2(x)\cot(y)dy = 0.$
- $\dfrac{e^y}{e^y+1}y' = \dfrac{1}{x}$, ◦ $3e^x\tan(y)dx + \dfrac{1-e^x}{\cos^2 y}dy = 0.$ $(*)$
- $y'\tan(x) = y$ $(*)$, ◦ $(x^2+1)y' = y^2+4.$ $(*)$

Exercise 03 : Integrate the following *homogeneous* differential equations :

- $y' = \dfrac{y}{x} - 1$, ◦ $y' = -\dfrac{x+y}{x}$ $(*)$ ◦ $(x-y)ydx - x^2dy = 0.$
- $(x^2+y^2)dx - 2xydy = 0$, $(*)$ ◦ $y' = \dfrac{x-y}{x+y}$

Exercise 04 : Find the general solution of following *linear* differential equations :

- $y' - \dfrac{y}{x} = x$, $(*)$ ◦ $(1-x^2)y' + xy = 2x.$ $(*)$
- $y' - y\cos(x) = \sin(2x).$

Exercise 05 : Find the particulars solutions satisfying the given initial conditions :

1. $xy' + y - e^x = 0; \quad y(a) = b;$ $(*)$
2. $y' - \dfrac{y}{1-x^2} - 1 - x = 0, \quad y(0) = 0.$
3. $y' - y\tan(x) = \dfrac{1}{\cos(x)}, \quad y(0) = 0.$ $(*)$

Exercise 06 : Find the general solution of *Bernoulli* differential equations :

- $y' + \dfrac{y}{x} = -xy^2$, ◦ $y'\sin x\cos x - 3y = -3y^{\frac{2}{3}}\sin^3 x$, $(*)$ ◦ $y' - y = 2\sqrt{y}e^{-x}.$

Exercise 07 : Integrate the following **second-order** differential equations :

1. $y'' + 3y' + 2y = 2x^2 + 8x + 7.$
2. $y'' - y = e^{2x} - e^x$
3. $y'' - 6y' + 9y = 2e^{3x} - e^x.$
4. $y'' + y = x\sin x.$ $(*)$
5. $y'' + 2y' + y = \cos^2 x.$ $(*)$
6. $y'' - 2y' + 2y = \sin(x).e^x$ $(*)$

(Home working)

1. Integrate the differential equation
$$-y' + y = x^2.$$
2. Find all the functions $z : \mathbb{R} \to]0, +\infty[$ which are derivable, solutions of differential equation
$$z' + z - x^2z^2 = 0$$
and such that $z(0) = \dfrac{1}{3}$.
- We can do the change variable $y = z^{-1}$.

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0.1 Solution

Exercise 01 : The type of first order differential equations :

1. $xy' = (x-1)y \Leftrightarrow \dfrac{dy}{dx} = \dfrac{x-1}{x}dx$ Separable equation
2. $(1+y^2)dy = xdx$, obvious
3. $2xyy' - y^2 + x = 0 \Leftrightarrow y - \dfrac{1}{2x}y = -\dfrac{1}{2}y^{-1}$ Bernoulli equation
4. $y' = \dfrac{x-y}{x+y} = \dfrac{1-\frac{y}{x}}{1+\frac{y}{x}}$ homogeneous equation
5. $y' - \dfrac{y}{1-x^2} = 1+x \Leftrightarrow y' - \dfrac{1}{1-x^2}y = 1+x$ linear equation

Exercise 02 :

$$(2): \quad \tan(x)\sin^2(y)dx + \cos^2(x)\cot(y)dy = 0.$$

We divided by $\sin^2(y)\cos^2(x)$ such that : $y \neq (2k+1)\dfrac{\pi}{2}$ and $x \neq k\pi$ So

$$(2) \Leftrightarrow \int \frac{\sin x}{\cos^3 x}dx + \int \frac{\cos y}{\sin^3 y}dy = 0,$$

after integration we obtain

$$(2) \Leftrightarrow \frac{1}{\cos^2 x} + \frac{1}{\sin^2 y} = C. \quad C \in \mathbb{R}.$$

$$(4): \quad 3e^x\tan(y)dx + \frac{1-e^x}{\cos^2 y}dy$$

$$(4) \Leftrightarrow \int \frac{3e^x}{1-e^x}dx = -\int \frac{dy}{\sin y \cos y}$$

separable equation.

$$-\int \frac{dy}{\sin y \cos y} = -\int \frac{1}{\tan y}\frac{1}{\cos^2 y}\,dy = -\int \frac{\frac{1}{\cos^2 y}}{\tan y}dy = -\ln|\tan y| + c$$

Then

$$3\ln|1-e^x| = \ln|\tan y| + C$$

$$\ln|\tan y| - 3\ln|1-e^x| = C$$

Finally

$$\tan y = C(1-e^x)^3$$

Exercise 03 :

1. $y' = \dfrac{y}{x} - 1$, we make the change of variable $\dfrac{y}{x} = t \Leftrightarrow y = tx \Leftrightarrow y' = t'x + t \Leftrightarrow \dfrac{dy}{dx} = \dfrac{dt}{dx}x + t$

   so $\quad y' = \dfrac{y}{x} - 1 \Leftrightarrow t'x = -1 \Leftrightarrow x = Ce^{-\frac{y}{x}}$.
2. $y' = -\dfrac{x+y}{x}$ the same work for this equation
3. $(x-y)ydx - x^2dy = 0 \Leftrightarrow \dfrac{dy}{dx} = \dfrac{y}{x} - (\dfrac{y}{x})^2$ the student must complete

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4. $(x^2+y^2)dx-2xydy=0 \Leftrightarrow \dfrac{dy}{dx}=\dfrac{1}{2}(\dfrac{1}{\frac{y}{x}}+\dfrac{y}{x})=F(\dfrac{y}{x})$ homogeneous equation make the change of variable as previous $y=tx$ next we resolve a separable equation.

5. $y'=\dfrac{x-y}{x+y}=\dfrac{1-\frac{y}{x}}{1+\frac{y}{x}} \Leftrightarrow \int\dfrac{dx}{x}=\int\dfrac{1+t}{1-2t-t^2}dt$ we have

$$\frac{1+t}{1-2t-t^2}=\frac{A}{t-1-\sqrt{2}}+\frac{B}{t-1+\sqrt{2}}$$

$$A=\frac{1+\sqrt{2}}{2}, \qquad B=\frac{1-\sqrt{2}}{2}$$

Then

$$\frac{1+t}{1-2t-t^2}=\frac{\frac{1+\sqrt{2}}{2}}{t-1-\sqrt{2}}+\frac{\frac{1-\sqrt{2}}{2}}{t-1+\sqrt{2}}$$

... Continue the solution

Exercise 04 : We chose

$$y'-y\cos(x)=\sin(2x).$$

— Solve without second member
$\dfrac{dy}{dx}=y\cos(x) \Leftrightarrow y=Ce^{\sin x}$

— with second member : by the variation constant method
$C'=\sin 2xe^{-\sin x} \Leftrightarrow C=e^{-\sin x}(-2\sin x-1)+K$ then $y=Ke^{\sin x}-2\sin x-1$

Exercise 05 :

$$y'-\frac{y}{1-x^2}-1-x=0, \quad y(0)=0.$$

For $x\in]-1,1[$

$$\frac{dy}{dx}=\frac{dx}{1-x^2} \Leftrightarrow y=Ce^{\arg thx}$$

by the variation constant method and after calculation we find

$$C=\frac{1}{2}\arcsin x+\frac{1}{4}\sin(2\arcsin x)+K$$

then the general solution given by

$$y=e^{\arg thx}(\frac{1}{2}\arcsin x+\frac{1}{4}\sin(2\arcsin x)+K)$$

So the solution satisfy the initial condition $y(0)=0$ is

$$y=e^{\arg thx}(\frac{1}{2}\arcsin x+\frac{1}{4}\sin(2\arcsin x))$$

Exercise 06 :

$$y'+\frac{y}{x}=-xy^2$$

Bernoulli equation we multiply by y^2 we obtain

$$y^{-2}y'+\frac{1}{x}y^{-1}=-x$$

Make the change of variable $y^{-1}=z \Leftrightarrow -y'y^{-2}=z'$ the Bernoulli equation become linear equation in the form

$$z'+\frac{1}{x}z=-x$$

We use the variation of constant method, the general solution of linear equation given by

$$z = \frac{K}{x} + \frac{x^2}{3}$$

, and the general solution of Bernoulli equation is

$$y = \frac{1}{z} = \frac{1}{\dfrac{K}{x} + \dfrac{x^2}{3}}$$

Exercise 07 :

$$y'' + 3y' + 2y = 2x^2 + 8x + 7..........(1)$$

— first the solution without second member The characteristic equation is

$$r^2 + 3r + 2 = 0 \Rightarrow r_1 = -1, \quad r_2 = -2$$

then

$$y_H = C_1 e^{-x} + C_2 e^{-2x}, \quad C_1, C_2, \quad \text{are constants}$$

— second with second member we know that $y_G = y_H + y_P$
The second member is polynomial then the particular solution

$$y_P = ax^2 + bx + c, \quad y_P' = 2ax + b, \quad y_P'' = 2a$$

We replace in (1) and by identification $a = 1, \quad b = \frac{5}{2}, \quad c = \frac{-5}{4}$ hence

$$y_P = x^2 + \frac{5}{2}x + \frac{-5}{4}$$

and

$$y_G = C_1 e^{-x} + C_2 e^{-2x} + x^2 + \frac{5}{2}x + \frac{-5}{4}$$

$$y'' - y = e^{2x} - e^{x}.......;(2)$$

Characteristic equation is

$$r^2 - 1 = 0, \quad r = \pm 1$$

Solution without second member is

$$C_1 e^{-x} + C_2 e^{x}$$

then we find y_P, use **the variation constant method** as follow : we put $y_1 = e^{-x}, \quad y_1 = e^{x}$ and solve the system equations

$$\begin{cases} C_1' y_1 + C_2' y_2 = 0 \\ C_1' y_1' + C_2' y_2' = f(x) \end{cases}$$

$$\begin{cases} C_1' e^{-x} + C_2' e^{x} = 0......(1) \\ -C_1' e^{-x} + C_2' e^{x} = f(x)...(2) \end{cases}$$

$$(1) + (2) : C_2' = \frac{1}{2}(e^x - 1) \Rightarrow C_2 = \frac{1}{2}e^x - \frac{x}{2} + K_2$$

$$(2) - (1) : C_1' = -\frac{1}{2}(e^{3x} - e^{2x}) \Rightarrow C_1 = -\frac{1}{6}e^{3x} + \frac{1}{4}e^{2x} + K_1$$

then

$$y_G = K_1 e^{-x} + K_2 e^{x} + \frac{1}{3}e^{2x} + \frac{1}{4}(1 - 2x)e^{x}$$

$$y'' - 6y' + 9y = 2e^{3x} - e^{x}......(3)$$

Characteristic equation is :

$$r^2 - 6r + 9 = 0 \Leftrightarrow (x-3)^2 = 0, \quad r = 3\text{double root}$$

so

$$y_H = (C_1 x + C_2)e^{3x}$$

For find the particular solution we can use

- First method : because the second member is the form $P(x)e^{\alpha x}$. We search $y_P = y_{P_1} + y_{P_2}$ such that $y_{P_1} = Cx^2 e^{\alpha x}, \quad \alpha = 3, \quad y_{P_2} = Ce^{x}$

$$y'_{P_1} = C(2x + 3e^{2x})e^{3x}, \quad y''_{P_1} = C(9x^2 + 12x + 2)e^{3x}$$

  After calculation we can find $C = 1$ then $y_{P_1} = x^2 e^{3x}$ in the same way, we find $y_{P_2} = -\frac{1}{4}e^{x}$ then

$$y_G = (C_1 x + C_2)e^{3x} + x^2 e^{3x} - \frac{1}{4}e^{x}$$

- we can use : the variation of constant method as previous

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