MOHAMED BOUDIAF UNIVERSITY OF MSILA

Faculty of Mathematics and Computer Science **Computer Science Department**

Module : Analysis 2 **University year :**2026/2027

Series N°2 : The Integrals

Exercise 01 : Compute the primitive functions :

$$\int (e^x)^{n+1}dx, \quad \int \frac{\cos(\ln x)}{x}dx, \quad \int \cos(x)\sin(x)dx, \quad \int \sin(2x)\cos(3x)dx.$$

$$\int \frac{e^x}{1+e^{2x}}dx, \quad \int \frac{dx}{x\sqrt{1-(\ln x)^2}}, \quad \int \frac{dx}{\sqrt{1+x^2}}, \quad \int sh(e^x)e^x dx$$

$$\int \frac{dx}{\sin x}dx, \quad \int \frac{1+x^5-x^6}{1-x}dx, \quad \int \frac{1}{\sqrt{x}-x}dx, \quad \int \frac{x^3-2}{x^3-x^2}dx.$$

Exercise 02 : Calculate the following integrals

$$I_1 = \int \arcsin(x)dx, \quad I_2 = \int x^n \ln(x)dx, \quad I_3 = \int_{-1}^{1} \sqrt{1-x^2}dx, \quad I_4 = \int \frac{xdx}{\sqrt{x^2+x+2}}$$

$$I_5 = \int \sqrt{1+x^2}dx, \quad I_6 = \int x\ln(\frac{1-x}{1+x})dx, \quad I_7 = \int_0^{\frac{\pi}{2}} \frac{\cos x}{(1+\sin^2 x)(1+\sin x)}dx.(\text{put } t=\sin x)$$

Exercise 03 : We consider

$$I_1 = \int_0^{\pi} (x\cos x)^2 dx; \quad I_2 = \int_0^{\pi} (x\sin x)^2 dx$$

- Find $I_1 + I_2$ and $I_1 - I_2$, and then deduce I_1, I_2.

Exercise 04 : Let the following function : $f(x) = \dfrac{1}{1+x^2}$

1. Find D_f, then calculate the area $S(\alpha)$ enclosed between the graph of f, the vertical lines $x=\alpha,\ x=-\alpha,\ (\alpha>0.)$ and horizontal line $y=0$.
2. Calculate $\lim\limits_{\alpha\to+\infty} S(\alpha)$

Exercise 05 : Let

$$F(x) = \int_1^x \cos(\ln t)dt; \quad G(x) = \int_1^x \sin(\ln t)dt$$

1. Using the integration by parts, show that

$$F(x) = x\cos(\ln x) - 1 + G(x), \quad G(x) = x\sin(\ln x) - F(x).$$

2. Deduce the expressions of $F(x)$ et $G(x)$.

Exercise 06 : Using the decomposition of rational fractions, calculate the following integrals :

$$\int_2^4 \frac{dx}{x(1+x^2)}, \quad \int_2^3 \frac{x^2-1}{(x-1)^2(x^2+1)}dx, \quad \int_2^4 \frac{x+1}{(x-\frac{1}{2})(x+\frac{1}{2})(x^2+1)}dx$$

Exercise 7 : (Home working 1) We consider the following integrals

$$K = \int_0^{\frac{\pi}{8}} e^{-2x}\cos(2x)dx, \quad I = \int_0^{\frac{\pi}{8}} e^{-2x}\cos^2(x)dx, \quad J = \int_0^{\frac{\pi}{8}} e^{-2x}\sin^2(x)dx,$$

— Calculate $K, \quad I+J$ et $I-J$.
— Deduce the values of I and J.

Exercise 08 : (Home working 2) Calculate the following integrals :

$$\int \frac{1}{\sqrt{4+x^2}}dx, \quad \int \frac{1}{\sqrt{x^2+3x-4}}dx, \quad \int \frac{(1-\sqrt{x^2+x+1})^2}{x^2\sqrt{x^2+x+1}}dx.$$

Dr : M. Touahria

Solution

Exercise 1 :

$$\int (e^x)^{n+1} dx = \int e^x (e^x)^n dx = \frac{1}{n+1}(e^x)^{n+1} + C$$

$$\int \frac{\cos(\ln x)}{x} dx,$$

we make change variable $\ln x = t \implies \frac{dx}{x} = dt$

$$\int \frac{\cos(\ln x)}{x} dx = \in \cos(t) dx = \sin(t) = \sin(\ln x) + C$$

$$\int \cos(x)\sin(x) dx = \frac{1}{2}\sin^2 x + C.$$

$$\int \sin(2x)\cos(3x) dx = \frac{1}{2}\int \sin(5x) - \sin(x) dx = -\frac{1}{10}\cos(5x) + \frac{1}{2}\cos(x) + C$$

By change variable $e^x = t \implies e^x dx = dt$

$$\int \frac{e^x}{1+e^{2x}} dx = \int \frac{t}{1+t^2} = \arctan(e^x) + C.$$

By the same way

$$\int \frac{dx}{x\sqrt{1-(\ln x)^2}} = \arcsin(\ln x) + C.$$

$$\int \frac{dx}{\sqrt{1+x^2}} = argsh(x) + C$$

$$\int sh(e^x) e^x dx = Ch(e^x) + C, \quad \text{make the change variable} \quad t = e^x.$$

$$\int \frac{dx}{\sin x} dx = \int \frac{dx}{\frac{1}{2}\sin\frac{x}{2}\cos\frac{x}{2}} dx = \int \frac{dt}{\sin t \cos t} = \int \frac{\frac{dt}{\cos^2 t}}{\tan t} = \ln|\tan(\frac{x}{2})| + C$$

$$\int \frac{1+x^5-x^6}{1-x} dx = \int \left(x^5 + \frac{1}{1-x}\right) dx = \frac{x^6}{6} - \ln|1-x| + C.$$

By change variable $\sqrt{x} = t \implies \frac{dx}{\sqrt{x}} = 2dt$

$$\int \frac{1}{\sqrt{x}-x} dx = \int \frac{2dt}{1-t} = -2\ln|1-\sqrt{x}| + C.$$

$$\int \frac{x^3-2}{x^3-x^2} dx = \int dx + \int \frac{x^2-2}{x^3-x^2} dx = \int dx + \int \frac{x+2}{x^2} dx - \int \frac{dx}{x-1}$$

$$= \int dx + 2\int \frac{dx}{x} + 2\int \frac{dx}{x^2} - \int \frac{dx}{x-1}$$

$$= x + C_1 + 2\ln|x| + C_2 - \frac{2}{x} + C_3 - \ln|x-1| + C_4$$

$$= x - \frac{2}{x} + \ln\frac{x^2}{|x-1|} + C, \quad C = C_1 + C_2 + C_3 + C_4$$

Exercise 02 :

By integration by parts

$$I_1 = \int \arcsin(x)dx = x\arcsin(x) + \sqrt{(1-x^2)} + C$$

$$I_2 = \int x^n \ln(x)dx = \frac{x^{n+1}}{n+1}\left(\ln x - \frac{1}{n+1}\right) + C.$$

$$I_3 = \int_{-1}^{1} \sqrt{1-x^2}dx, \quad \text{we make change variable } x = \sin t$$

$$I_3 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t dt = \left[\frac{1-\cos(2t)}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0.$$

$$I_4 = \int \frac{xdx}{\sqrt{x^2+x+2}} = \int \frac{(2x+1)dx}{\sqrt{x^2+x+2}} - \frac{1}{2}\int \frac{dx}{\sqrt{x^2+x+2}} = 2\sqrt{x^2+x+2} + C_1 - \frac{1}{2}\int \frac{dx}{\sqrt{(x+1/2)^2+7/4}}$$

So, first we make the change variable $t = x + 1/2 \implies dt = dx$

$$\int \frac{dx}{\sqrt{(x+1/2)^2+7/4}} = \int \frac{dx}{\sqrt{t^2+7/4}} = \int \frac{2dt}{\sqrt{7}\sqrt{(\frac{2t}{\sqrt{7}})^2+1}}$$

second, We make another change variable $\dfrac{2t}{\sqrt{7}} = u \implies dt = \sqrt{7}/2du$, thus

$$\int \frac{2dt}{\sqrt{7}\sqrt{(\frac{2t}{\sqrt{7}})^2+1}} = \int \frac{du}{\sqrt{u^2+1}} = argsh(u) + C_2$$

then

$$I_4 = 2\sqrt{x^2+x+2} - \frac{1}{2}argsh\left[\frac{2}{\sqrt{7}}(x+\frac{1}{2})\right].$$

We make the change variable $x = sh(t) \implies dx = ch(t)dt$

$$I_5 = \int \sqrt{1+x^2}dx = \int ch^2(t)dt = \int \frac{1}{4}(e^{2t} - e^{-2t} + 2)dt = \frac{1}{4}(\frac{e^{2t}-e^{-2t}}{2} + 2t) + C$$

$$= \frac{1}{4}sh(2t) + \frac{t}{2} + C = \frac{1}{4}(2sh(t)ch(t) + \frac{t}{2} + C = \frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}argsh(x) + C$$

$$\frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\ln(x+\sqrt{1+x^2}) + C.$$

By integration by parts we obtain

$$I_6 = \int x\ln(\frac{1-x}{1+x})dx = \frac{x^2}{2}\ln\left[\frac{1-x}{1+x}\right] + \int(-1 + \frac{1}{1-x^2})dx$$

$$= \frac{x^2}{2}\ln\left[\frac{1-x}{1+x}\right] - x + argth(x) + C = \frac{x^2}{2}\ln\left[\frac{1-x}{1+x}\right] - x + \frac{1}{2}\ln\left[\frac{1+x}{1-x}\right] + C.$$

By the change variable, we obtain

$$I_7=\int_0^{\frac{\pi}{2}}\frac{\cos x}{(1+\sin^2 x)(1+\sin x)}dx=\int_0^1\frac{dt}{(1+t^2)(1+t)}dt$$

we have

$$\frac{1}{(1+t^2)(1+t)}=\frac{A}{1+t}+\frac{Bt+C}{1+t^2}$$

By identification, we find

$$\boxed{A=C=\frac{1}{2},\quad B=-\frac{1}{2}}$$

Then

$$I_7=\left[\frac{1}{2}\ln|1+t|-\frac{1}{4}\ln|1+t^2|+\frac{1}{2}\arctan t\right]_0^1=\boxed{\frac{1}{4}\ln 2+\frac{\pi}{8}}$$

Exercise 03 : We have

$$I_1+I_2=\int_0^\pi x^2dx=\left[\frac{x^3}{3}\right]_0^\pi=\frac{\pi^3}{3}$$

$$I_1-I_2=\int_0^\pi x^2\cos^2 xdx$$

By doing twice integration by parts, we obtain

$$I_1-I_2=\frac{\pi}{2}.$$

So

$$\left\{I_1+I_2=\tfrac{\pi^3}{3}I_1-I_2=\tfrac{\pi}{2}\right.$$

Therefore

$$\boxed{I_1=\frac{\pi^3}{6}+\frac{\pi}{4},\quad I_2=\frac{\pi^3}{3}-\frac{\pi}{4}}$$

Exercise 04 :

1. $D_f=\mathbb{R}$.
2. $$S(\alpha)=\int_{-\alpha}^{\alpha}(f(x)-y)dx=\int_{-\alpha}^{\alpha}\frac{dx}{1+x^2}=[\arctan(x)]_{-\alpha}^{\alpha}=2\arctan(\alpha)$$
3. $\lim_{\alpha\to+\infty}S(\alpha)=\lim_{\alpha\to+\infty}2\arctan(\alpha)=2(\frac{\pi}{2})=\pi$.

Exercise 05 :

Using the integration b parts

$$F(x)=[t\cos(\ln t)]_1^x-\int_1^x t(-\frac{1}{t}\sin(\ln t))dt=x\cos(\ln x)-1+G(x)$$

Also

$$G(x)=x\sin(\ln x)-F(x)$$

Then the expressions of $F(x)$ and $G(x)$ are :

$$F(x)=\frac{x}{2}(\cos(\ln x)+\sin(\ln x))-\frac{1}{2}.$$

$$G(x)=x\sin(\ln x)-\frac{x}{2}(\cos(\ln x)+\sin(\ln x))+\frac{1}{2}.$$

$$G(x)=\frac{x}{2}(\sin(\ln x)-\cos(\ln x))+\frac{1}{2}.$$

Exercise 06 :

$$\int_2^4 \frac{dx}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2}$$

By identification, we obtain $A = 1, \quad B = -1, \quad C = 0$

Thus

$$\int_2^4 \frac{dx}{x(1+x^2)} = [\ln x - \frac{1}{2}\ln(x^2+1)]_2^4 = \frac{2\sqrt{5}}{\sqrt{17}}.$$

$$\int_2^3 \frac{x^2-1}{(x-1)^2(x^2+1)} dx,$$

$$\frac{x^2-1}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}.$$

After expansion and identification of coefficients, we obtain

$$A = -\frac{1}{2}, \qquad B = 1, \qquad C = \frac{1}{2}, \qquad D = -\frac{1}{2}.$$

$$\int_2^4 \frac{x+1}{(x-\frac{1}{2})(x+\frac{1}{2})(x^2+1)} dx$$

we decompose :

$$\frac{x+1}{(x-\frac{1}{2})(x+\frac{1}{2})(x^2+1)} = \frac{A}{x-\frac{1}{2}} + \frac{B}{x+\frac{1}{2}} + \frac{Cx+D}{x^2+1}.$$

After computation :

$$A = \frac{6}{5}, \quad B = -\frac{2}{5}, \quad C = -\frac{4}{5}, \quad D = -\frac{4}{5}.$$

Thus

$$I_3 = \int_2^4 \left(\frac{6/5}{x-\frac{1}{2}} - \frac{2/5}{x+\frac{1}{2}} - \frac{4x/5}{x^2+1} - \frac{4/5}{x^2+1} \right) dx.$$

$$I_3 = \frac{6}{5}\ln\left|x-\frac{1}{2}\right| - \frac{2}{5}\ln\left|x+\frac{1}{2}\right| - \frac{2}{5}\ln(1+x^2) - \frac{4}{5}\arctan x \Big|_2^4.$$

Dr : M. Touahria