MOHAMED BOUDIAF UNIVERSITY OF MSILA

Faculty of Mathematics and Computer Science **Computer Science Department**

Module : Analysis 2 **University year :**2026/2027

Series N°1 : The finite expansion

Exercise 01 :

1. Using Taylor formula. Calculate the finite expansion of $3-rd$ order in the neighborhood of $x_0 = 0$ of the following functions :

$$f_1(x) = \ln(1+x), \quad f_2(x) = e^x, \quad f_3(x) = \cos x, \quad f_4(x) = \sin x, \quad f_5(x) = \tan x,$$
$$f_6(x) = x^5 + x^4 + x^3 + x^2 + x + 1, \quad f_7(x) = \frac{1+x}{2+x}, \quad f_8(x) = \frac{1}{\cos x},$$
$$f_9(x) = sh(x), \quad f_{10}(x) = \arcsin x, \quad f_{11}(x) = th(x)$$

2. Deduce the finite expansion of $3-rd$ order around to $x_0 = 0$ of the following functions :

$$g_1(x) = \cos x + \sin x, \quad g_2(x) = e^x.\tan x, \quad g_3(x) = \frac{\ln(1+x)}{x}, \quad g_4(x) = e^{\cos x}$$
$$g_5(x) = \sin(\ln(1+x)), \quad g_6(x) = \sqrt{\cos x}, \quad g_7(x) = e^{\arcsin x}$$

Exercise 02 :

Using the euclidean division, Calculate the finite expansion of $4-th$ order around to $x_0 = 0$ of the following functions :

$$f_1(x) = \frac{\ln(1+x)}{\cos x}, \quad f_2(x) = \frac{e^x}{1+x-x^2}$$

Exercise 03 :

Using the finite expansion, calculate the following limits

$$\lim_{x\to 0} \frac{\sin x}{x}, \quad \lim_{x\to 0} \frac{1-\cos x}{x^2}, \quad \lim_{x\to 0} \frac{\sin(\cos(x)-1)}{x},$$
$$\lim_{x\to 0} \frac{1 - x\sin x - \cos x}{(e^x-1)^2}, \qquad \lim_{x\to 0} \frac{1}{(\sin x)^4}[\sin \frac{x}{1-x} - \frac{\sin x}{1-\sin x}]$$

Exercise 04 :

Calculate the following finites expansions

1) $\frac{1}{x}$ to 3-rd order at $x_0 = 2$, 2) $\ln x$ to 4-th order at $x_0 = 2$

3) $\cos x$ to 3-rd order at $x_0 = \frac{\pi}{3}$, 4) $\sqrt{x}$ to 3-rd order at$x_0 = 2$

Exercise 05 :

Calculate the following limits such that a, b two positives real.

$$\lim_{x\to+\infty} \left[\frac{\ln(1+x)}{\ln(x)}\right]^x, \qquad \lim_{x\to+\infty} \left[\frac{a^{\frac{1}{x}} + b^{\frac{1}{x}}}{2}\right]^x$$

Exercise 06 :**Homework.** Let f a function defined by

$$f(x) = x(1 + e^{\frac{1}{x}}).$$

1. Calculate the finite expansion of f in the neighborhood to ∞ of degree 2.
2. Deduce that the graph of f admits the line of equation $y = 2x + 1$ as asymptote, then study the relative position between the graph of f and this asymptote.

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Solutions

Exercise 01 :

1. Taylor expansions at $x_0 = 0$ of 3^{rd} order

We recall the standard Taylor formula around 0 is Mac-Laurin polynomials :

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + ... + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + O(x-x_0)^n$$

Such that $f'(x), f''(x), ..., f^n(x)$ are successively derivatives functions of f. **<u>Particular case. Mac-Laurin</u>** if $x = 0$.

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + ... + \frac{f^{(n)}(0)}{n!}x^n + O(x^n)$$

We can replace the remainder part $O(x-x_0)^n$ by $(x-x_0)^n\epsilon(x)$ with $\epsilon(x) \longrightarrow 0$ when $x \longmapsto x_0$.

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + x^3\epsilon(x), \quad \lim_{x\to 0}\epsilon(x) = 0. \quad e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + x^3\epsilon(x),$$

$$\cos x = 1 - \frac{x^2}{2} + x^3\epsilon(x), \quad \sin x = x - \frac{x^3}{6} + x^3\epsilon(x), \quad \tan x = x + \frac{x^3}{3} + x^3\epsilon(x).$$

$$f_1(x) = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + x^3\epsilon(x)$$

$$f_2(x) = e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + x^3\epsilon(x)$$

$$f_3(x) = \cos x = 1 - \frac{x^2}{2} + x^3\epsilon(x)$$

$$f_4(x) = \sin x = x - \frac{x^3}{6} + x^3\epsilon(x)$$

$$f_5(x) = \tan x = x + \frac{x^3}{3} + x^3\epsilon(x)$$

Polynomial :

$$f_6(x) = 1 + x + x^2 + x^3 + x^4 + x^5$$

f_7 is rational function :

$$f_7(x) = \frac{1+x}{2+x} = \frac{1}{2} + \frac{x}{4} - \frac{x^2}{8} + \frac{x^3}{16} + x^3\epsilon(x)$$

$$f_8(x) = \frac{1}{\cos x} = 1 + \frac{x^2}{2} + x^3\epsilon(x)$$

$$f_9(x) = shx = x + \frac{x^3}{6} + x^3\epsilon(x)$$

$$f_{10}(x) = x + \frac{x^3}{6} + x^3\epsilon(x)$$

$$f_{11}(x) = thx = x - \frac{x^3}{3} + x^3\epsilon(x)$$

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2. Deduction of expansions (Using the operations properties of finite expansions)

$$g_1(x) = \cos x + \sin x = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + x^3\epsilon(x)$$

$$g_2(x) = e^x \tan x = x + x^2 + \frac{5}{6}x^3 + x^3\epsilon(x)$$

$$g_3(x) = \frac{\ln(1+x)}{x} = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + x^3\epsilon(x)$$

$$g_4(x) = e^{\cos x} = e^{1-\frac{x^2}{2}} = e\left(1 - \frac{x^2}{2}\right) = e - e\frac{x^2}{2} + x^3\epsilon(x)$$

$$g_5(x) = \sin(\ln(1+x)) = (x - \frac{x^2}{2} + \frac{x^3}{3} + x^3\epsilon(x)) - \frac{(x - \frac{x^2}{2} + \frac{x^3}{3} + x^3\epsilon(x))^3}{6} = x - \frac{x^2}{2} + \frac{x^3}{6} + x^3\epsilon(x)$$

$$g_6(x) = \sqrt{\cos x} = 1 - \frac{x^2}{4} + x^3\epsilon(x), \quad \lim_{x\to 0}\epsilon(x) = 0.$$

$$g_7(x) = e^{\arcsin x} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + x^3\epsilon(x)$$

Exercise 02 : Using euclidean division by increasing power order

$$\frac{\ln(1+x)}{\cos x} = \frac{\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\right)}{\left(1 + \frac{x^2}{2} + \frac{5x^4}{24}\right)}$$

$$= x - \frac{x^2}{2} + \frac{5x^3}{6} - \frac{x^4}{4} + x^4\epsilon(x)$$

Is the same way from this function

$$\frac{e^x}{1+x-x^2} = 1 + x + \frac{3}{2}x^2 + \frac{13}{6}x^3 + \frac{37}{24}x^4 + x^4\epsilon(x)$$

Exercise 03 : Using Taylor expansions :

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{x + x\epsilon(x)}{x} = 1$$

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{1 - (1 - \frac{x^2}{2} + x^2\epsilon(x))}{x^2} = \frac{1}{2}$$

$$\lim_{x\to 0}\frac{\sin(\cos x - 1)}{x} = 0$$

$$\lim_{x\to 0}\frac{1 - x\sin x - \cos x}{(e^x - 1)^2} = -\frac{1}{2}$$

After calculation and simplification we can find

$$\sin\left(\frac{x}{1-x}\right) = x + x^2 + \frac{5}{6}x^3 + \frac{1}{2}x^4 + x^4\epsilon(x)$$

$$\frac{\sin x}{1-\sin x} = x + x^2 + \frac{5}{6}x^3 + \frac{2}{3}x^4 + x^4\epsilon(x)$$

Thus

$$\lim_{x\to 0} \frac{1}{(\sin x)^4}\left[\sin\left(\frac{x}{1-x}\right) - \frac{\sin x}{1-\sin x}\right] = \frac{-1}{6}$$

Exercise 04 :

Using change variable : $x - x_9 = t \implies x = t + x_0$

$$\frac{1}{x} = \frac{1}{2+t} = \frac{1}{2(1+\frac{t}{2})} = \frac{1}{2}\left(1 - \frac{t}{2} + \frac{t^2}{4} - \frac{t^3}{8}\right)$$

$$= \frac{1}{2} - \frac{t}{4} + \frac{t^2}{8} - \frac{t^3}{16} + t^3\epsilon(t) \quad \lim_{t\to 0}\epsilon(t) = 0.$$

$$= \frac{1}{2} - \frac{x-2}{4} + \frac{(x-2)^2}{8} - \frac{(x-2)^3}{16} + (x-2)^3\epsilon(x-2), \quad \lim_{x\to 2}\epsilon(x-2) = 0.$$

Use the same technique from the rest

$$\ln x = \ln 2 + \frac{x-2}{2} - \frac{(x-2)^2}{8} + \frac{(x-2)^3}{24} - \frac{(x-2)^4}{64} + (x-2)^4\epsilon(x-2)$$

$$\cos x = \frac{1}{2} - \frac{\sqrt{3}}{2}(x - \tfrac{\pi}{3}) - \frac{1}{4}(x-\tfrac{\pi}{3})^2 + \frac{\sqrt{3}}{12}(x-\tfrac{\pi}{3})^3 + (x-\tfrac{\pi}{3})^3\epsilon(x-\tfrac{\pi}{3})$$

$$\sqrt{x} = \sqrt{2} + \frac{x-2}{2\sqrt{2}} - \frac{(x-2)^2}{8\sqrt{2}} + \frac{(x-2)^3}{16\sqrt{2}} + (x-2)^3\epsilon(x-2)$$

Exercise 05 :

1.

$$\lim_{x\to+\infty}\left(\frac{\ln(1+x)}{\ln x}\right)^x = \lim_{x\to+\infty} e^{\ln(x(\frac{\ln(1+x)}{\ln x}))}$$

We have

$$\frac{\ln(1+x)}{\ln x} = \frac{\ln x + \ln(1+1/x)}{\ln x} = 1 + \frac{\ln(1+1/x)}{\ln x}$$

$\ln(1+1/x) = 1/x + 1/x.\epsilon(1/x)$ near to $+\infty$

$$\frac{\ln(1+x)}{\ln x} = 1 + \frac{1}{x\ln x} + \frac{1}{x\ln x}\epsilon(\frac{1}{x}). \quad \lim_{t\to 0}\epsilon(\frac{1}{x}) = 0.$$

$$e^{x(\frac{\ln(1+x)}{\ln x})} = e^{x\ln(1+\frac{1}{x\ln x}+\frac{1}{x\ln x}\epsilon(\frac{1}{x}))}$$

$$\lim_{x\to\infty}\left(\frac{\ln(1+x)}{\ln x}\right)^x = \lim_{x\to\infty} e^{\frac{1}{\ln x}+\frac{1}{\ln x}\epsilon(\frac{1}{x})} = 1$$

2. $\lim\limits_{x\to+\infty}\left(\frac{a^{1/x}+b^{1/x}}{2}\right)^x = \lim\limits_{x\to+\infty} e^{x\ln}\left(\frac{a^{1/x}+b^{1/x}}{2}\right) = \lim\limits_{t\to 0} e^{1/t\ln}\left(\frac{a^t+b^t}{2}\right)$

Around to 0, we have

$a^t = 1 + t\ln a + t\epsilon(t),\ b^t = 1 + t\ln b + t\epsilon(t)$

so

$$\lim_{t\to 0} e^{1/t\ln}\left(\frac{a^t+b^t}{2}\right) = \lim_{t\to 0} e^{1/t\ln\left(\frac{2+t(\ln a+\ln b)}{2}+t\epsilon(t)\right)} = e^{\frac{1}{2}\ln(ab)} = \sqrt{ab}.$$

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