MOHAMED BOUDIAF UNIVERSITY OF MSILA
Faculty of Mathematics and Computer Science
Computer Science Department
University year :2025/2026

1.st year bachelor's degree - Semester 2
Final Exam in Analysis 2
Date : 13/05/2026 Duration : 2 h

Course questions : (03 Points)

1. Write the Mac-Laurin polynomial with remainder of degree n.
2. State the linearity and Shales relation proprieties of Riemann integral.
3. Give a sufficient condition for existence the primitives of a function.

Exercise 1 : (06 Points)

1. Find the finite expansion near to 0 of the function $x \mapsto \dfrac{1}{1-\sin x}$ of degree 4.
2. Find the finite expansion near to 0 of the function $x \mapsto \dfrac{1}{1-\arctan x}$ of degree 4.
3. Deduce the finite expansion at 0 of the function f defined by the relation
$$f(x) = \frac{1}{1-\sin x} - \frac{1}{1-\arctan x} \quad \text{of degree 4.}$$
4. Calculate $\lim\limits_{x\to 0}\left[\dfrac{f(x)}{x^3}\right]$.

We mention : $\boxed{\arctan x = x - \dfrac{x^3}{3} + x^4\epsilon(x), \quad \sin x = x - \dfrac{x^3}{6} + x^4\epsilon(x)}$

Exercise 2 : (05 Points)

1. Decompose into simple fractions : Find A, B and; C such that :
$$\frac{3}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$
2. Calculate $I = \displaystyle\int \frac{3}{x^3+1}\,dx$.
3. Deduce $J = \displaystyle\int_0^1 \frac{3}{x^3+1}\,dx$.

Exercise 3 : (06 Points)

We consider the differential equation
$$y' + y\tan x = \sin x \cos x. \tag{1}$$

1. Determine the type of equation(1).
2. Solve the homogeneous equation (without second member) associated with (1).
3. Use the constant of variation method to find the particular solution of (1) then give all solutions of (1).
4. Calculate the solution of (1) satisfying $y\left(\dfrac{\pi}{4}\right) = 0$.

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SOLUTION OF SECOND EXAM

Course questions : (03 Points)

1. Mac-Laurin polynomial

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + ... + \frac{f^{(n)}(0)}{n!}x^n + x^n\epsilon(x), \quad \lim_{x\to 0}\epsilon(x) = 0. \qquad (01)$$

2. Linearity and Chasles relation :

$$\int_a^b (\lambda f + \mu g)(x)dx = \lambda \int_a^b f(x)dx + \mu \int_a^b g(x)dx. \qquad (0.5)$$

$$\forall c \in [a, b] : \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx. \qquad (0.5)$$

3. The sufficient condition is the continuous. (01)

Exercise 1 : (06 Points)

1. The finite expansion of $x \mapsto \dfrac{1}{1-\sin x}$:

   We can proceed in two ways : either by composition using the Taylor expansion of : $t \mapsto \dfrac{1}{1-t}$ at 0, or by use the euclidean division according to increasing power order.

   We have

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + t^4 + t^4\epsilon(t) \quad \lim_{t\to 0}\epsilon(t) = 0. \qquad (0.5)$$

and

$$\sin x = x - \frac{x^3}{6} + x^4\epsilon(x),$$

Then

$$\frac{1}{1-\sin x} = 1 + \left(x - \frac{x^3}{6}\right) + \left(x - \frac{x^3}{6}\right)^2 + \left(x - \frac{x^3}{6}\right)^3 + \left(x - \frac{x^3}{6}\right)^4 + x^4\epsilon(x). \qquad (01)$$

$$\frac{1}{1-\sin x} = 1 + x + x^2 + \frac{5}{6}x^3 + \frac{2}{3}x^4 + x^4\epsilon(x). \qquad (0.5)$$

2. The finite expansion at 0 of $x \mapsto \dfrac{1}{1-\arctan x}$:

   We have

$$\arctan x = x - \frac{x^3}{3} + x^4\epsilon(x),$$

$$\begin{array}{c|c} 1 & 1 - x + \frac{x^3}{3} \\ \hline x - \frac{x^3}{3} & 1 + x + x^2 + \frac{2}{3}x^3 + \frac{1}{4}x^4 \\ x^2 - \frac{x^3}{3} - \frac{x^4}{3} & \\ \frac{2}{3}x^3 - \frac{x^4}{3} & \\ \frac{1}{4}x^4 & \\ 0 & \end{array} \qquad (1.5)$$

$$\frac{1}{1-\arctan x} = \frac{1}{1 - x + \frac{x^3}{3}} = 1 + x + x^2 + \frac{2}{3}x^3 + \frac{1}{4}x^4 + x^4\epsilon(x). \qquad (0.5)$$

3. Finally we deduce the finite expansion of f

$$f(x) = \frac{1}{6}x^3 + \frac{1}{3}x^4 + x^4\epsilon(x). \qquad (01)$$

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4. The limit

$$\lim_{x\to 0}\frac{f(x)}{x^3}=\lim_{x\to 0}\left(\frac{1}{6}+\frac{1}{3}x\right)=\boxed{\frac{1}{6}}. \qquad (01)$$

Exercise 2 :(06 Points)

1. Partial fraction decomposition

   After identification, we obtain

$$\boxed{A=1, \qquad B=-1, \qquad C=2.} \qquad (1,5)$$

   That is,

$$\frac{3}{x^3+1}=\frac{1}{x+1}+\frac{-x+2}{x^2-x+1}. \qquad (0.5)$$

2. Let

$$I=\int\frac{3}{x^3+1}\,dx.$$

   Then

$$I=\int\left(\frac{1}{x+1}+\frac{-x+2}{x^2-x+1}\right)dx=\int\frac{1}{x+1}dx+\int\frac{-x+2}{x^2-x+1}dx.$$

   So

$$\int\frac{dx}{x+1}=\ln|x+1|+C_1. \qquad (0.5)$$

   We write

$$\int\frac{-x+2}{x^2-x+1}=-\frac{1}{2}\int\frac{2x-1}{x^2-x+1}dx+\frac{3}{2}\int\frac{1}{x^2-x+1}dx.$$

   We have

$$-\frac{1}{2}\int\frac{2x-1}{x^2-x+1}dx=-\frac{1}{2}\ln(x^2-x+1)+C_2, \qquad (0.5)$$

   and

$$\int\frac{1}{x^2-x+1}dx,$$

   by change variable we obtained arctan form. Like this :

$$\int\frac{1}{x^2-x+1}dx=\int\frac{dx}{\frac{3}{4}\left(\frac{2x-1}{\sqrt{3}}\right)^2+1},\ (0.5)\text{use the change variable}\, t=\frac{2x-1}{\sqrt{3}} \implies dx=\frac{\sqrt{3}}{2}dt\,(0.5)$$

   We obtain

$$\frac{2}{\sqrt{3}}\int\frac{dt}{t^2+1}=\frac{2}{\sqrt{3}}\arctan t=\frac{2}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right)+C_3 \qquad (0.5)$$

   Finally

$$I=\ln|x+1|-\frac{1}{2}\ln(x^2-x+1)+\sqrt{3}\arctan\left(\frac{\sqrt{3}}{3}(2x-1)\right)+C. \quad C=C_1+C_2+C_3 \qquad (0.5)$$

3. Hence,

$$I=\int_0^1\frac{3}{x^3+1}\,dx=\boxed{\ln 2+\frac{\pi\sqrt{3}}{3}} \qquad (01)$$

Exercise 3 :(06 Points)

1. The type is First order linear differential equation. (0.5)
2. The associated homogeneous equation is

$$y' + y\tan x = 0 \qquad (0.25)$$

So

$$\frac{dy}{y} = -\frac{\sin x}{\cos x}dx \iff \int \frac{dy}{y} = \int -\frac{\sin x}{\cos x}dx \qquad (0.5)$$

the solution is

$$\boxed{y_h = C\cos x}. \qquad (0.5)$$

where C is an arbitrary constant.

3. Using the method of variation of constants :

$$y = C(x)\cos x \iff y' = C'(x)\cos x - C(x)\sin x. \qquad (0.5)$$

Substituting y and y' into (1), (05) we obtain

$$C'(x) = \sin x, \quad (0.25) \text{ by integration } \quad C(x) = -\cos x + K. \qquad (0.5)$$

So, replace $C(x)$ in $y = (-\cos x + K)\cos x = -\cos^2 x + K\cos x$

Hence, a particular solution of (1) is

$$\boxed{y_P = -\cos^2 x \quad \text{or } y_P = -\frac{1}{2}\cos(2x) - \frac{1}{2}}. \qquad (01)$$

Therefore, the general solution of (1) is

$$\boxed{y_G = K\cos x - \cos^2 x = K\cos x - \frac{1}{2}\cos(2x) - \frac{1}{2}}. \qquad (0.5)$$

4. The solution satisfy the condition$y\left(\frac{\pi}{4}\right) = 0$

$$y_G\left(\frac{\pi}{4}\right) = 0 \iff K = \frac{\sqrt{2}}{2}. \qquad (0.5)$$

Thus, the required solution is

$$\boxed{y = \frac{\sqrt{2}}{2}\cos x - \cos^2 x}. \qquad (0.5)$$

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