MOHAMED BOUDIAF UNIVERSITY OF MSILA
Faculty of Mathematics and Computer Science
Computer Science Department
University year :2024/2025

1.st year bachelor's degree - Semester 2
Final Exam in Analysis 2
Date : 14/05/2025 Duration : 1 h 30 m

Course questions : (03 Pt)

1. **What conditions must a function f satisfy in order to admit a Taylor polynomial approximation with remainder of degree 3 near to 0 ? Then write this polynomial.**
2. **Recall the technique of integration by parts for a definite integral.**

Exercise 1 : (06 Pt)

1. **Write the finite expansion of the following functions in the neighborhood to 0 of degree 4.**
$$x \mapsto e^x, \quad x \mapsto \ln(1+x), \quad x \mapsto \cos(x), \quad x \mapsto \sin(x).$$
2. **Find the finite expansion at 0 of the function $x \mapsto e^{\sin x}$ of the degree 2.**
3. **Find the finite expansion at 0 of the function $x \mapsto \ln[\cos(6x)]$ of the degree 4.**
4. **Using the finite expansion, calculate the limit**
$$\lim_{x \to 0} \left(\frac{x^2 e^{\sin x}}{\ln[\cos(6x)]} \right).$$

Exercise 2 : (07 Pt)

Let
$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos x + \sin x} dx, \quad \text{and} \quad J = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\cos x + \sin x} dx.$$

1. **Without calculating I and J. Show that $I = J$.**
2. **Verify that :** $\forall x \in \mathbb{R} : \cos x + \sin x = \sqrt{2} \cos(x - \frac{\pi}{4})$.
3. **Deduce that**
$$I + J = \frac{\sqrt{2}}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{\cos x}.$$
4. **Calculate $I + J$, then deduce the value of I and J.**

$\boxed{\textbf{Indication :} \rightrightarrows \quad \tan(\frac{\pi}{8}) = \sqrt{2} - 1.}$

Exercise 3 : (04 Pt)

Give the type then integrate the following differential equation.
$$6y' = 2y - xy^4.$$

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SOLUTION OF SECOND EXAM25 : Analysis 02

Course questions : (03 Pt)

1. **The conditions that a function f satisfy : $f'(0), f''(0), f^3(0)$ exists..(0,5)**

$$f(x) = f(0)+f'(0)x+\frac{f''(0)}{2}x^2+\frac{f^{(3)}(0)}{6}+x^3\epsilon(x), ...(0,5) \qquad \lim_{x\to 0}\epsilon(x) = 0...(0,5)$$

2. **Let u and v two functions of the class $\mathcal{C}^1$ i.e (u', v' exists and continuous) on the interval $[a,b]$....$(0,75)$ Then**

$$\int_a^b u(x)v'(x)dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)dx...(0,75)$$

Exercise 1 :

1. **The finite expansion of f in the neighborhood of 0 at order 3.**
We have

$$e^x = 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+x^4\epsilon(x), \qquad (0,5)$$

$$\ln(x+1) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+x^4\epsilon(x), \qquad (0,5)$$

$$\sin x = x-\frac{x^3}{6}+x^4\epsilon(x), \qquad (0,5)$$

$$\cos x = 1-\frac{x^2}{2}+\frac{x^4}{24}+x^4\epsilon(x), \qquad (0,5)$$

2. **The finite expansion at 0 of the function $x \mapsto e^{\sin x}$ of the degree 2.**

$$e^{\sin x} = 1+(x-\frac{x^3}{6})+\frac{(x-\frac{x^3}{6})^2}{2!} = 1+x+\frac{x^2}{2}+x^2\epsilon(x)....(1,5)$$

3. **The finite expansion at 0 of the function $x \mapsto \ln[\cos(6x)]$ of $4-th$. degree**

First, we have

$$\cos(6x) = 1-\frac{(6x)^2}{2}+\frac{(6x)^4}{24} = 1-18x^2+54x^4+x^4\epsilon(x)....(0,5)$$

Then

$$\ln(\cos(6x)) = \ln(1+(-18x^2+54x^4)) = (-18x^2+54x^4)-\frac{(-18x^2+54x^4)}{2}+...$$

$$= -18x^2-108x^4+x^4\epsilon(x)....(01)$$

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4. **Calculate the limit**

$$\lim_{x\to 0}\left(\frac{x^2 e^{\sin x}}{\ln[\cos(6x)]}\right)=\lim_{x\to 0}\frac{1+x+\dfrac{x^2}{2}}{-18-108x^2}=-\frac{1}{18}...(01)$$

<u>**Exercise 2**</u> **: (07 Pt)**

Let

$$I=\int_0^{\frac{\pi}{2}}\frac{\cos^2 x}{\cos x+\sin x}dx,\quad \textbf{and}\quad J=\int_0^{\frac{\pi}{2}}\frac{\sin^2 x}{\cos x+\sin x}dx.$$

1. **We Show that** $I=J$. **We observe that**

$$I-J=\int_0^{\frac{\pi}{2}}\frac{\cos^2 x-\sin^2 x}{\cos x+\sin x}dx=\int_0^{\frac{\pi}{2}}(\cos x-\sin x)dx=[\sin x+\cos x]_0^{\frac{\pi}{2}}=0.\quad (01)$$

Thus $I=J$.

2. **We have**

$$\sqrt{2}\cos(x-\frac{\pi}{4})=\sqrt{2}(\cos x\cos(\frac{\pi}{4})+\sin x\sin(\frac{\pi}{4}))=\sqrt{2}(\frac{\sqrt{2}}{2}\cos x+\frac{\sqrt{2}}{2}\sin x).$$

$$\cos x+\sin x.\qquad (0,5)$$

3. **Let us also note that**

$$I+J=\int_0^{\frac{\pi}{2}}\frac{\cos^2 x+\sin^2 x}{\cos x+\sin x}dx=\int_0^{\frac{\pi}{2}}\frac{dx}{\cos x+\sin x}$$

$$\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{2}\cos(x-\frac{\pi}{4})}.\qquad (0.5)$$

Put $t=x-\frac{\pi}{4}$. **Then** $dt=dx$. **...(0,25)**

Also $x=0\Rightarrow t=-\frac{\pi}{4}$ **and** $x=\frac{\pi}{2}\Rightarrow t=\frac{\pi}{4}....(0,25)+(0,25)$

Hence

$$I+J=\frac{\sqrt{2}}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{dt}{\cos t}=\frac{\sqrt{2}}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{dx}{\cos x}.\qquad (0.5)$$

4. **Now for Calculating** $I+J$, **we make the change variable** $u=\tan(\frac{x}{2})$.

then we have

$$\cos x=\frac{1-\tan^2(\frac{x}{2})}{1+u^2(\frac{x}{2})}=\frac{1-u^2}{1+u^2},\quad dx=\frac{2du}{1+u^2}...(0,5)$$

The terminals : For $x=-\frac{\pi}{4}\Rightarrow u=\tan(-\frac{\pi}{8});...(0,25)$

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and for $x=\dfrac{\pi}{4} \Rightarrow u=\tan(\dfrac{\pi}{8})...(0,25)$
Knowing that

$$\tan(\frac{\pi}{8})=\sqrt{2}-1, \qquad \tan(-\frac{\pi}{8})=1-\sqrt{2}.$$

Hence

$$I+J=\frac{\sqrt{2}}{2}\int_{1-\sqrt{2}}^{\sqrt{2}-1}\frac{\frac{2du}{1+u^2}}{\frac{1-u^1}{1+u^2}}=\sqrt{2}\int_{1-\sqrt{2}}^{\sqrt{2}-1}\frac{du}{1-u^2}. \qquad (01)$$

Using the method of decomposition into simple elements. We have

$$\frac{1}{1-u^2}=\frac{1}{(1-u)(1+u)}=\frac{A}{1-u}+\frac{B}{1+u}$$

who gives by identification $A=B=\dfrac{1}{2}...(0,5)$ **Therefore**

$$\sqrt{2}\int_{1-\sqrt{2}}^{\sqrt{2}-1}\frac{du}{1-u^2}=\frac{\sqrt{2}}{2}\left(\int_{1-\sqrt{2}}^{\sqrt{2}-1}\frac{du}{1-u}+\int_{1-\sqrt{2}}^{\sqrt{2}-1}\frac{du}{1+u}\right)$$

$$\frac{\sqrt{2}}{2}\left[-\ln|1-u|+\ln||1+u|\right]_{1-\sqrt{2}}^{\sqrt{2}-1}$$

$$=\sqrt{2}(\ln\sqrt{2}+\ln(2-\sqrt{2}))=\sqrt{2}\ln(\frac{\sqrt{2}}{2-\sqrt{2}})=\sqrt{2}\ln(1+\sqrt{2}). \qquad (0.5)$$

We deduce I **and** J. **We have**

$$\begin{cases} I-J=0 \\ I+J=\sqrt{2}\ln(1+\sqrt{2})... \qquad (0,25) \end{cases}$$

Then $I=J=\ln(1+\sqrt{2}) \qquad (0.5)$

Exercise 3 : (04 Pt)

$$6y'=2y-xy^4 \Leftrightarrow 6y'-2y=xy^4....(E1)$$

The type of this equation is Bernoulli differential equation. because $\alpha=4\neq 0,$ **and** 1 **...(01)**
We multiply (E1) by y^{-4}, **we obtain**

$$6y'y^{-4}-2y^{-3}=x....(E2)$$

We make the change variable $z=y^{-3}\Rightarrow z'=-3y'y^{-4}$
The equation (E2) becomes

$$-2z'-2z=x....(E3)$$

Is linear equation. We solve it as follow

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- **The homogeneous solution** $y_H = Ce^{-x}...(01)$ **solution of homogeneous equation** $-2z' - 2z = 0$
- **search a particular solution either by constant variation method or by observe that the particular solution** z_P **is a first degree polynomial.** $z_P = az + b \Rightarrow z'_P = a$ **we replace** z_P **and** z'_P **in (E3), we obtain**

$$a = -\frac{1}{2}, \quad b = \frac{1}{2}.$$

Then the general solution of (E3) is

$$z_G = y_H + y_P = Ce^{-x} - \frac{1}{2}x + \frac{1}{2}...(01)$$

Then The general solution of (E1) is

$$y^{-3} = z \Leftrightarrow y^{-3} = Ce^{-x} - \frac{1}{2}x + \frac{1}{2}$$

Finally

$$y = (Ce^{-x} - \frac{1}{2}x + \frac{1}{2})^{-\frac{1}{3}}...(01)$$

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