Functions of Several Variables

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Introduction

When studying functions of a single variable, we noticed that the analysis of many phenomena requires the use of functions of two or more independent variables. Let us give some examples.

Examples

1. The area S of a rectangle with sides x and y is given by the well-known formula
$$S = xy.$$
To each pair values of x and y corresponds a well-determined value of the area S. Thus S is a function of two variables.

2. The volume V of a rectangular parallelepiped, whose edge lengths are respectively x, y, z, is given by the formula
$$V = xyz.$$
Here V is a function of three variables x, y, z.

3. $$u = \frac{x^2 + y^2 + z^2 + t^2}{\sqrt{1 + x^2}}.$$
Here u is a function of four variables.

Function of two variables

Definition

If to each pair x, y of values of two independent variables x and y, taken in a certain domain of definition D, there corresponds a well-determined value of the variable z, then z is called a function of two variables x and y defined in the domain D.

A function of two variables is denoted by

$$z = f(x, y) \quad \text{or} \quad z = F(x, y), \quad \text{etc.}$$

Domain of definition

Definition

The domain of existence of a function of two variables can be interpreted geometrically as follows : if each pair of values x and y is represented by a point $M(x, y)$ in the plane Oxy, then the domain of definition of the function is represented by a set of points in this plane. We shall call this set of points the domain of definition of the function. In particular, this domain may occupy the entire plane Oxy.

Domain of definition

- The domain of definition that we consider will consist of portions of the plane bounded by certain curves.
- The curve that bounds the domain of definition is called the boundary of the domain.
- Points of the domain that do not belong to the boundary are called interior points of the domain.
- Domain consisting only of interior points is called an open domain.
- A domain completed by its boundary is called a **closed domain**.
- A domain is said to be bounded if there exists a constant C such that the distance OM from any point M of this domain to the origin O is less than C, that is

$$|OM| < C.$$

Example

Determine the natural domain of definition of the function

$$z = 2x - y.$$

The analytic expression $2x - y$ is defined for all arbitrary values of x and y. Consequently, the natural domain of definition of this function coincides with the entire plane Oxy. On other world

$$\boxed{D = \mathbb{R}^2}$$

Examples

Example

$$z = \sqrt{1 - x^2 - y^2}.$$

For z to be real, the radical must be non-negative ; in other words, x and y must satisfy the inequalities

$$1 - x^2 - y^2 \geq 0 \quad \text{or} \quad x^2 + y^2 \leq 1.$$

The set of points $M(x, y)$ whose coordinates satisfy this inequality is the part of the plane bounded by the circle of radius 1, centered at the origin (more precisely, the interior of this circle together with its circumference).

$$\boxed{D = \{(x, y)\} \in \mathbb{R}^2 : x^2 + y^2 \leq 1}$$

Examples

Example

$$z = \log(x + y).$$

Logarithms are defined only for positive numbers ; therefore one must necessarily have

$$x + y > 0 \quad \text{or} \quad y > -x.$$

The natural domain of definition of this function is therefore the half-plane located above the line

$$y = -x$$

(the points of the line do not belong to the domain).

Several variables functions

It is easy to extend the definition of a function of two real independent variables to the case of three or more independent variables
Let $n \in \mathbb{N}^*$. We make the convention that $\mathbb{R}^n$ denotes the set of ordered n-tuples $(x_1, x_2, ..., x_n)$ of real numbers $x_i \in \mathbb{R}, \quad i = 1, ..., n$.

$$\mathbb{R}^n = \{(x_1, x_2, ..., x_n) : x_i \in \mathbb{R}, \quad i = 1, ..., n\}.$$

The first appointment of this course is to study the generalization of the notion of continuity and differentiability for functions of several variables. In our course we will focus on functions f defined on a set D, generally open in $\mathbb{R}^3$ or $\mathbb{R}^2$, with values in $\mathbb{R}$.

Distance and Norm

Definition

Let $n \in \mathbb{N}^*$. A distance on $\mathbb{R}^n$ is any function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+$ satisfying the following properties :

1. $d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, y) \leq d(x, z) + d(z, y)$.

The last property is called the triangle inequality or convexity

Remark

The notion $d(x, y)$ is read as the distance from $xtoy$.

Example

Example

1. $d(x, y) = |x - y|$ is a distance on $\mathbb{R}$.
2. Let $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+$ defined by

$$d(x, y) = \begin{cases} 1 & \text{if} x \neq y, \\ 0 & \text{if} x = y. \end{cases}$$

Then d is a distance on $\mathbb{R}^n$, called the **discrete distance**.

Properties

Definition

A norm on $\mathbb{R}^n$ is all application

$$N : \mathbb{R}^n \to \mathbb{R}^+$$
$$x \mapsto N(x),$$

such that $\forall \lambda \in \mathbb{R}, \forall x, y \in \mathbb{R}^n$

- $N(x) = 0 \iff x = 0.$
- $N(\lambda x) = \lambda N(x).$
- $N(x + y) \leq N(x) + N(y).$

The Norm

Remark

If $n = 1$, N is the absolute value $|.|$. Henceforth, we will adopt the notation $\|.\|$ instead of $N(.)$, which is often found in the literature (books, etc.) $\|.\|$ is read as "the norm of x.

Let $x \in \mathbb{R}^n$, $x = (x_1, x_2, ..., x_n)$. We define :

- $\|x\|_1 = |x_1| + |x_2| + ... + |x_n|$
- $\|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + ... + |x_n|^2}$
- $\|x\|_\infty = \sup\{|x_i| : 1 \leq i \leq n\}$

These are three norms of $\mathbb{R}^n$.

Remark

For any $a(x_0, y_0, z_0) \in \mathbb{R}^3$ and $r > 0$, we denote by $B_{\mathbb{R}^3}(a, r)$ the open ball (respectively $\overline{B}_{\mathbb{R}^3}(a, r)$ the closed ball) of $\mathbb{R}^3$ that is the set of all $a(x, y, z) \in \mathbb{R}^3$ such that

$$\|(x, y, z) - (x_0, y_0, z_0)\| < r \quad (\text{respectively} \leq r).$$

Definition

A numerical function of n variables is each application f defined as follow :

$$f : \mathbb{R}^n \to \mathbb{R}$$

$$(x_1, x_2, ..., x_n) \mapsto f(x_1, x_2, ..., x_n).$$

We called domain of definition of f noted by D_f the set $x \in \mathbb{R}^n$ for which this function is defined.

Examples

For example the function f defined by

$$\bullet \quad f : \mathbb{R}^2 \to \mathbb{R}$$

$$(x, y) \mapsto f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$$

$D_f = \mathbb{R}^2$.

$$\bullet \quad f : \mathbb{R}^3 \to \mathbb{R}$$

$$(x, y, z) \mapsto x^{yz}$$

$D_f = \{(x, y, z) \in \mathbb{R}^3 : x > 0\}$.

$$\bullet \quad f : \mathbb{R}^2 \to \mathbb{R}$$

$$(x, y) \mapsto \sqrt{1 - x^2 - y^2}$$

$D_f = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. Represent the the sphere (or ball) with center is $(0, 0, 0)$ and a radius equal 1.

The graph

Remark

If $n = 2$, the graph of function f is a subset of $\mathbb{R}^3$ defined by

$$G_f = \{(x, y, z) \in \mathbb{R}^3 : z = f(x, y)\}.$$

- The graph of the function of two variables has dimension 3.
- If $n \geq 3$ the graph of function of several variables $f : \mathbb{R}^3 \to \mathbb{R}$ can always be defined, but can't represented neither even imagined.

Limit of function at point x_0

The concept of the limit of functions of several variables corresponds naturally to that of functions of a single real variable.
Unilateral limits, that is left and right hand limits, lose their meaning and are replaced by the many possible directional limits.

Definition

Let $f : \mathbb{R}^n \to \mathbb{R}$ a function. We will say that f has a limit l when x tends to x_0 if

$$\forall \epsilon > 0, \exists \delta > 0 : \|x - x_0\| < \delta \implies |f(x) - l| < \epsilon.$$

In this case we will say that

$$\lim_{x \to x_0} f(x) = l$$

Limit of function at point x_0

Remark

We consider f function of two reals variables

1. The limit if it exists is unique.
2. If f has a limit l, then the restriction of f to any curve (in particular straight lines) passing through $x_0 = (a, b)$ has the same limit.

Attention : If the restriction to every straight line passing through $x_0 = (a, b)$ has the same limit, cannot conclude that the limit exists.

Trick : In practice, to show that a function of several variables does not admit a limit at $x_0 = (a, b)$ it's enough to explicitly a restriction to a curve passing trough x_0 that does not admit a limit, or two restrictions that lead to two different limits. But to prove the existence of a limit, one must consider the general case.

Example

Show in two different ways that :

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

does not exists.

Example : first way

- First way : two straight lines

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{y\to 0} \frac{-y^2}{y^2} = -1 \qquad \text{along the direction } x = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{y\to 0} \frac{x^2}{x^2} = 1 \qquad \text{along the direction } y = 0$$

Example : second way

○ Second way : Polar coordinates

We make the change variable around $x_0 = (a, b)$ with $r > 0$ and $\theta \in [0, 2\pi[$

$$\begin{cases} x - a = r\cos\theta \\ y - b = r\sin\theta \end{cases} \implies \begin{cases} x = a + r\cos\theta \\ y = b + r\sin\theta \end{cases}$$

then

$$\lim_{(x,y)\to(a,b)} f(x, y) = \lim_{r\to 0} f(a + r\cos\theta, b + r\sin\theta).$$

Thus

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{r\to 0} \frac{r^2\cos 2\theta}{r^2} = \cos 2\theta. \quad \text{The limit varies according to } \theta$$

Then the limit does not exists.

Continuity of a function

Definition

We say that a function f is continuous at $x_0 \in \mathbb{R}^n$ if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

We will say that f is continuous on $D \subset \mathbb{R}^n$ if its continuous at all points of D

Example

The function f defined by

$$(x, y) \mapsto f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$$

Is not continuous at $(0, 0)$ because does not limit at this point.

Indeed

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{y\to 0} \frac{0}{y} = 0 \qquad \text{along the direction } x = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{y\to 0} \frac{x^2}{2x^2} = \frac{1}{2} \qquad \text{along the direction } y = x$$

We should continuity on all directions.

Continuity

Exercise

Let f a function defined on $\mathbb{R}^2$ by

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^4 - 2x^2y + 3y^2}, & \text{if} \quad (x,y) \neq (0,0) \\ 0, & \text{if} \quad (x,y) = (0,0). \end{cases}$$

Show that the restriction of f on all straight line passing through the point $(0,0)$ is continuous. But is not continuous at the origin.

Exercise : Continuity

- The restriction of f on x-axis and y-axis is the zero function. Therefore the restriction on straight lines $y = mx, \quad m \neq 0$ is the function of the form

$$f(x,y) = f(x,mx) = \begin{cases} \dfrac{mx}{x^2 - 2mx + 3m^2}, & \text{if} \quad x \neq 0 \\ 0, & \text{if} \quad x = 0. \end{cases}$$

$$\lim_{x \to 0} \frac{mx}{x^2 - 2mx + 3m^2} = 0 = f(0).$$

Then the restriction of f on all straight line passing through the origin is continuous.

- But if we consider the restriction of f on the curve $y = x^2$. We have

$$f(x,y) = f(x,x^2) = \frac{x^4}{2x^4} = \frac{1}{2}.$$

Thus, $f(x,x^2)$ does not tends to zero when x tends to zero. Then f is not continuous at the origin.

Partial derivatives

Definition

(First partial derivatives) Let f function of two variables defined over $D \subset \mathbb{R}^2$ and $(x_0, y_0) \in D$. The first partial derivatives of f at (x_0, y_0) are the derivatives of partial functions $f_x,\ f_y$ evaluates at (x_0, y_0) in other words

$$f'_y = \frac{\partial f}{\partial x}(x_0, y_0) = \lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}.$$

Is the derivative of f to respect to x at the point (x_0, y_0).

$$f'_x = \frac{\partial f}{\partial y}(x_0, y_0) = \lim_{y \to y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0} = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}.$$

Is the derivative of f to respect to y at the point (x_0, y_0).

Example : First partial derivatives

Example

We consider the function f defined by

$$f(x,y) = 3 - x^2 - 2y^2$$

Calculate the first derivatives function of f at $x_0 = (1,1)$.

$$\frac{\partial f}{\partial x}(1,1) = \lim_{h\to 0} \frac{f(1+h,1) - f(1,1)}{h} = \lim_{x\to 1} \frac{f(x,1) - f(1,1)}{x-1} = -2$$

$$\frac{\partial f}{\partial y}(1,1) = \lim_{h\to 0} \frac{f(1,1+h) - f(1,1)}{h} = \lim_{y\to 1} \frac{f(1,y) - f(1,1)}{y-1} = -4$$

Derivative : Example

Example

If $f(x,y) = x^3 + y^2 \sin x$, then

$$\frac{\partial f}{\partial x}(x,y) = 3x^2 + y^2 \cos x, \qquad \frac{\partial f}{\partial y}(x,y) = 2y \sin x.$$

Example

If $g(x,y,z) = \arctan(xy^2) + e^z$, then

$$\frac{\partial g}{\partial x}(x,y,z) = \frac{y^2}{1+x^2y^4},$$

$$\frac{\partial g}{\partial y}(x,y,z) = \frac{2xy}{1+x^2y^4},.$$

$$\frac{\partial g}{\partial z}(x,y,z) = e^z.$$

Second derivatives

Lat $(x_0, y_0) \in \mathbb{R}^2$ and f be a function that admits partial derivatives in a neighborhood of (x_0, y_0).

$$\nabla_{(x_0,y_0)} : (x, y) \mapsto \begin{cases} f'_x(x, y) \\ f'_y(x, y). \end{cases}$$

Assume that f'_x and f'_y also admit partial derivatives at the point (x_0, y_0). Then we can define four partial derivatives :

$$f''_{xx}(x_0, y_0) = f''_{x^2}(x_0, y_0) = \frac{\partial^2 f}{\partial x^2}(x_0, y_0), \qquad f''_{yx}(x_0, y_0) = \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0).$$

$$f''_{yy}(x_0, y_0) = f''_{y^2}(x_0, y_0) = \frac{\partial^2 f}{\partial y^2}(x_0, y_0), \qquad f''_{xy}(x_0, y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0).$$

Second derivatives

Definition

These partial derivatives are called **second-order partial derivatives.**

Example

f function defined by $f(,y) = x^y, \quad x > 0$. We have

$$\frac{\partial^2 f}{\partial y \partial x}(x, y) = \frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{x^y}{x}(1 + x^y).$$

check

Second derivatives

Example

Let $f : \mathbb{R}^2 \to \mathbb{R}$ function such that :

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}, & \text{if} \quad (x,y) \neq (0,0) \\ 0, & \text{if} \quad (x,y) = (0,0). \end{cases}$$

The first partial derivatives are continuous except at $(0,0)$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = 1, \quad \text{but } \frac{\partial^2 f}{\partial x \partial y}(x,y) = -1.$$

Schwartz theorem

Theorem

(Schwartz) Let f function of two variables with values in $\mathbb{R}$, admits second partial derivatives f''_{xy} and f''_{yx} continuous in a neighborhood of (x_0, y_0). Then

$$\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) = \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0).$$

Exercise

Exercise 4 : Series 04

Let f function defined by

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

Calculate

1. $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$.
2. $\dfrac{\partial f}{\partial x}(x,y)$ and $\dfrac{\partial f}{\partial y}(x,y)$ at $(x,y) \neq (0,0)$.
3. $\dfrac{\partial^2 f}{\partial x \partial y}(0,0)$ and $\dfrac{\partial^2 f}{\partial y \partial x}(0,0)$.
4. What do you conclude ?

Solution

1.
$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{y - 0} = 0$$
$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y - 0} = 0$$

2.
$$\frac{\partial f}{\partial x}(x,y) = \frac{y^3(y^2 - x^2)}{(x^2 + y^2)^2}$$
$$\frac{\partial f}{\partial x}(x,y) = \frac{3x^3y^2 + xy^4}{(x^2 + y^2)^2}$$

3.
$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y})(0,0) = \lim_{x\to 0} \frac{\frac{\partial f}{\partial y}(x,0) - \frac{\partial f}{\partial y}(0,0)}{x - 0} = 0,$$
$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x})(0,0) = \lim_{x\to 0} \frac{\frac{\partial f}{\partial x}(x,0) - \frac{\partial f}{\partial x}(0,0)}{x - 0} = 1,$$

Solution

Then, we conclude that :

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$$

And the second partial derivatives are not continuous at $(0,0)$. Schwartz theorem

Exercise 1 : Series 04

Give the domain of definition of the following functions then represent it geometrically.

$$f(x,y) = \frac{1}{x} - \frac{1}{y+1}, \quad g(x,y) = \sqrt{1-xy}, \quad h(x,y) = \ln(xy),$$

$$e(x,y) = \ln[(x^2+y^2-4)(9-x^2-y^2)]$$

Exercise 2 : Series 04

Verify if the following functions are continuous at $(0,0)$

$$(x,y) \mapsto f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

$$(x,y) \mapsto g(x,y) = \begin{cases} (x^2+y^2)\sin \dfrac{1}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

$$(x,y) \mapsto h(x,y) = \begin{cases} \dfrac{(x+y)^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

End Chapter 04