Differential equations

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Differential equation

A **differential equation** is a mathematical equation that relates an unknown function $y = f(x)$ to one or more of its derivatives $y', y'', y^{(3)}, ..., y^{(n)}$. By the relations as

$$G(x, y, y', y'', y^{(3)}, ..., y^{(n)}) = 0, \quad \text{or} \quad y' = F(x, y, y'', y^{(3)}, ..., y^{(n)}).$$

It describes how a quantity changes with respect to another variable and is widely used to model many phenomena in physics, engineering, biology, and economics.
In general, if $y = f(x)$ we symbolize with

$$y' = \frac{dy}{dx}, \quad y'' = \frac{d^2y}{dx^2}, \quad y^{(3)} = \frac{d^3y}{dx^3}, ..., y^{(n)} = \frac{d^ny}{dx^n}.$$

First order differential equations

Example 1

For example,

$$\frac{dy}{dx} = 3x^2$$

is a differential equation stating that the rate of change of y with respect to x is $3x^2$.

Definition 2

An equation of the form

$$G(x, y, y') = 0 \iff y' = F(x, y);$$

where y is a function of x is the unknown, its called **first order differential equation.**

Separable equation

They are of the form

$$f(y)y' = g(x). \tag{1}$$

The general solution is given by

$$\int f(y)dy = \int g(x)dx, \quad \text{because } y' = \frac{dy}{dx}.$$

For example we consider the first order differential equation

$$y' - 4y = 0 \Leftrightarrow \int \frac{dy}{y} = \int 4dx$$

$$\boxed{y = C.e^{4x}, \quad C \in \mathbb{R}} \qquad \textit{check}$$

Separable equations

Example 3

Integrate the following separable equations:

1. $(y^2 + 1)y' = x + 1.$

2. $(x - 1)y' + \sqrt{1 - y^2} = 0.$

Separable equation: Solution

$$\textbf{1.}\ (y^2+1)y' = x+1$$

First, we write the equation in differential form:

$$(y^2+1)\frac{dy}{dx} = x+1$$

Separate the variables:

$$(y^2+1)\,dy = (x+1)\,dx$$

Integrate both sides:

$$\int (y^2+1)\,dy = \int (x+1)\,dx \iff \frac{y^3}{3}+y = \frac{x^2}{2}+x+C$$

Hence the implicit solution is $\boxed{\frac{y^3}{3}+y = \frac{x^2}{2}+x+C}$

Separable equation: Solution

$$(x-1)\frac{dy}{dx} = -\sqrt{1-y^2}$$

Separate the variables:

$$\frac{dy}{\sqrt{1-y^2}} = -\frac{dx}{x-1}$$

Integrate both sides:

$$\int \frac{dy}{\sqrt{1-y^2}} = -\int \frac{dx}{x-1}$$

$$\arcsin(y) = -\ln|x-1| + C$$

or equivalently

$$\boxed{\arcsin(y) + \ln|x-1| = C \iff y = \sin(C - \ln|x-1|)\,.}$$

Homogeneous equations

They are of the form:

$$y' = F(\frac{y}{x}). \tag{2}$$

The resolution (or integration) of this type tends to the resolution of **separable equations**, we set

$$\frac{y}{x} = t \Leftrightarrow y = tx,$$

then

$$y' = xt' + t \iff \frac{dy}{dx} = x\frac{dt}{dx} + t.$$

The general solution is given by

$$\boxed{x = C.e^{\int \frac{dt}{F(t)-t}}}$$

Homogeneous equations

For example, we solve the equation

$$xy' - 2y = x \implies y' = 2(\frac{y}{x}) + 1,$$

Use the substitution

$$y = tx \implies y' = t + x\frac{dt}{dx}$$

By substitute we obtain:

$$x\frac{dt}{dx} = 2t + 1$$

Separate variables:

$$\frac{dt}{2t+1} = \frac{dx}{x}$$

Integrate:

$$\frac{1}{2}\ln|2t+1| = \ln|x| + C$$

Homogeneous equations

Substitute $t = \frac{y}{x}$:

$$2\frac{y}{x} + 1 = Cx^2$$

Multiply by x:

$$2y + x = Cx^3$$

Thus the general solution is

$$\boxed{y = \frac{Cx^3 - x}{2}}$$

Homogeneous equations

Example 4

Integrate the following homogeneous equation:

$$x^2ydx - y^3dy = x^3dy$$

Homogeneous equations

First rewrite the equation:

$$x^2y\,dx = (x^3 + y^3)\,dy$$

$$\frac{dy}{dx} = \frac{x^2y}{x^3 + y^3}$$

Since the equation is homogeneous, use the substitution

$$y = tx \qquad \Rightarrow \qquad \frac{dy}{dx} = t + x\frac{dt}{dx}.$$

By substitute we obtain:

$$t + x\frac{dt}{dx} = \frac{x^2(tx)}{x^3 + (tx)^3} = \frac{t}{1 + t^3}.$$

Homogeneous equations

Thus

$$x\frac{dt}{dx} = \frac{t}{1+t^3} - t = t\left(\frac{1}{1+t^3} - 1\right) = -\frac{t^4}{1+t^3}.$$

Separate the variables:

$$\int \frac{1+t^3}{t^4}\,dt = -\int \frac{dx}{x}.$$

By integration, we find:

$$-\frac{1}{3t^3} + \ln|t| = -\ln|x| + C.$$

Substitute $t = \dfrac{y}{x}$:

$$-\frac{1}{3\left(\frac{y}{x}\right)^3} + \ln\left|\frac{y}{x}\right| = -\ln|x| + C.$$

Homogeneous equations

$$-\frac{x^3}{3y^3} + \ln\left|\frac{y}{x}\right| = -\ln|x| + C.$$

Finally the solutions are

$$\boxed{\ln|y| - \frac{x^3}{3y^3} = C}.$$

Linear differential equations

They are in the form

$$y' + f(x)y = g(x), \tag{3}$$

where f, g are continuous functions on some interval I in $\mathbb{R}$, $g(x)$ is called the right hand side of the equation.

$\star$ **Integrate homogeneous equation (i.e., Without the second member)**

$$y' + f(x)y = 0,$$

it is a separable equation. The solutions are given by

$$\boxed{y_H = Ce^{-\int f(x)dx}, \quad C \in \mathbb{R}}$$

Linear differential equations

⋆ **Integrate with second member (or, complete equation.)**
The general solution of the complete equation (3)

$$\boxed{y_G = y_H + y_P,}$$

where y_G : is the general solution of (3), y_H : the homogeneous solution of (3) without second member and y_P : is the particular solution of (3).
Question: How to find y_P?

- By observation with the naked eye. Else
- By the variation of constant method.

Linear differential equations

We put:

$$y_0(x) = C(x).y_1(x), \text{where} \quad y_1(x) = e^{-\int f(x)dx}.$$

So

$$y_0'(x) = C'(x)e^{-\int f(x)dx} - C(x)f(x)e^{-\int f(x)dx}.$$

Then we substitute in (3), we find

$$C'(x) = g(x)e^{\int f(x)dx} \Rightarrow C(x) = \int g(x)e^{\int f(x)dx}\,dx$$

then, we obtain

$$y_0(x) = \left(\int g(x)e^{\int f(x)dx}dx\right)e^{-\int f(x)dx}$$

Linear differential equations

Example 5

We integrate the linear equation

$$xy' - 2y = x.$$

- With the naked eye, we can see that $-x$ is particular solution. Homogeneous solution (or, solution without second member)

$$xy' - 2y = 0 \Rightarrow y = C.x^2. \quad \text{Then} \quad y_G = C.x^2 - x, \quad C \in \mathbb{R}.$$

- By the constant variation method, we have

$$y' = C'.x^2 + 2x.C \quad \text{thus } C' = \frac{1}{x^2} \Rightarrow C = -\frac{1}{x} + K.$$

By substitute C in $y = C.x^2$. We obtain The general solution of complete equation

$$y_G = Kx^2 - x, \quad K \in \mathbb{R}.$$

Linear differential equations

Exercise

Solve (or, integrate) the linear equation

$$(1 - x^2)y' - xy = x.$$

Solution

First divide the equation by $1-x^2$:

$$y' - \frac{x}{1-x^2}y = \frac{x}{1-x^2}.$$

1. Solution of the associated homogeneous equation

$$y' - \frac{x}{1-x^2}y = 0.$$

$$\frac{y'}{y} = \frac{x}{1-x^2}.$$

Integrate:

$$\int \frac{dy}{y} = \int \frac{x}{1-x^2}dx.$$

$$\int \frac{x}{1-x^2}dx = -\frac{1}{2}\ln|1-x^2|.$$

Solution

Thus

$$\ln|y| = -\frac{1}{2}\ln|1-x^2| + C$$

$$y = \frac{C}{\sqrt{1-x^2}}.$$

2. Integrate the complete equation. Variation of the constant.
Let $C = C(x)$. Then

$$y = \frac{C(x)}{\sqrt{1-x^2}}.$$

Compute the derivative:

$$y' = \frac{C'(x)}{\sqrt{1-x^2}} + C(x)\frac{x}{(1-x^2)^{3/2}}. \quad \textit{check}$$

Substitute y and y' into complete equation

Solution

$$y' - \frac{x}{1-x^2}y = \frac{x}{1-x^2}.$$

$$\frac{C'(x)}{\sqrt{1-x^2}} + C(x)\frac{x}{(1-x^2)^{3/2}} - \frac{x}{1-x^2}\frac{C(x)}{\sqrt{1-x^2}} = \frac{x}{1-x^2}.$$

The terms containing $C(x)$ cancel, hence

$$\frac{C'(x)}{\sqrt{1-x^2}} = \frac{x}{1-x^2}.$$

Thus

$$C'(x) = \frac{x}{\sqrt{1-x^2}}.$$

Integrate:

$$C(x) = \int \frac{x}{\sqrt{1-x^2}}dx.$$

Solution

$$C(x) = -\sqrt{1-x^2} + C_1.$$

3. General solution

$$y = \frac{C(x)}{\sqrt{1-x^2}} = \frac{-\sqrt{1-x^2} + C_1}{\sqrt{1-x^2}}.$$

$$y = -1 + \frac{C_1}{\sqrt{1-x^2}}.$$

$$\boxed{y_G(x) = \frac{C}{\sqrt{1-x^2}} - 1}.$$

Bernoulli equations

They are of the form

$$y' + f(x)y = y^{\alpha} g(x). \tag{4}$$

- $y = 0,$ is a particular solution.
- $\alpha = 0,$ complete linear equation.
- $\alpha = 1,$ linear equation without second member.
- $\alpha \neq 0,\ \alpha \neq 1,\ y \neq 0,$ Bernoulli equation.

Bernoulli equations

Solving this equation is done as follows
We multiply both side equation (4) by $y^{-\alpha}$, we obtain

$$y^{-\alpha}y' + y^{1-\alpha}f(x) + = g(x).$$

we make the change variable

$$z = y^{1-\alpha} \Rightarrow z' = (1-\alpha)y^{-\alpha}y'.$$

Hence

$$\frac{z'}{1-\alpha} + f(x)z = g(x).$$

We are in the case of a linear equation.

Bernoulli equations

Example 6

$$xy' - y = y^2 \ln x. \quad (x > 0)$$

Is Bernoulli equation, $\alpha = 2$. We multiply both side previous equation by y^{-2}

$$xy^{-2}y' - y^{-1} = \ln x. \quad \text{Put } z = y^{-1} \Rightarrow z' = -y^2 y',$$

we obtain

$$z' + \frac{1}{x} z = \frac{\ln x}{x}. \quad \text{complete linear differential equation}$$

Bernoulli equations

1. Solution of the associated homogeneous equation

$$z' + \frac{1}{x} z = 0$$

By integration

$$\int \frac{dz}{z} = -\int \frac{dx}{x} \iff \ln \frac{z}{C} = -\ln x \iff z = \frac{C}{x}.$$

2. Variation of the constant method

$$z = \frac{C(x)}{x}.$$

By derivation

$$z' = \frac{C'(x)}{x} - \frac{C(x)}{x^2}.$$

Substitute z and z' into the original equation

Bernoulli equations

$$\left(\frac{C'(x)}{x}-\frac{C(x)}{x^2}\right)+\frac{1}{x}\frac{C(x)}{x}=\frac{\ln x}{x}.$$

Simplify:

$$\frac{C'(x)}{x}=\frac{\ln x}{x}.$$

Then

$$C(x)=\int \ln x\,dx=x\ln x-x+C_1.$$

Therefore the General solution is

$$z=\frac{C(x)}{x}=\frac{x\ln x-x+C_1}{x}=\ln x-1+\frac{C_1}{x}.$$

$$\boxed{y(x)=\frac{1}{\ln x-1+\frac{C}{x}}}.$$

Riccati equations

A first order differential equations are said to be Riccati equations if they are of the form

$$A(x)y' + B(x)y + C(x)y^2 = D(x), \tag{5}$$

where A, B, C and D are real continuous functions of real variable α being assumed to be non-zero. To solve this equation we must first find a particular solution y_P.

Riccati equations

$$A(x)y_P' + B(x)y_P + C(x)y_P^2 = D(x),$$

We make the change variable

$$t = y - y_P \implies y = t + y_P \implies y' = t' + y_P'$$

and $y^2 = t^2 + 2y_P t + y_P^2$.

$$A(x)(t' + y_P') + B(x)(t + y_P) + C(x)(t^2 + 2y_P t + y_P^2) = D(x)$$

$$\implies A(x)t' + [B(x) + 2C(x)y_P]t + C(x)t^2 = 0,$$

and this last equation is Bernoulli equation in t with $\alpha = 2$, an equation that we know how to solve.

Riccati equations

Example 7

Solve this Riccati equation

$$e^{-x}y' - 2e^{x}y + y^2 = 1 - e^{2x}, \tag{6}$$

where the particular solution $y_P = e^x$.

Example

We assume

$$t = y - e^x \implies y = t + e^x \implies y' = t' + e^x$$

and

$$y^2 = t^2 + 2e^x t + e^{2x}.$$

Replacing in (6) we obtain;

$$e^{-x}(t' + e^x) - 2e^x(y = t + e^x) + (t^2 + 2e^x t + e^{2x}) = 1 - e^2 x,$$

so

$$e^{-x}t' + t^2 = 0 \iff \int \frac{dt}{t^2} = \int -e^x dx,$$

Example

by integration, we find

$$-\frac{1}{t} = -e^{x} + C, \quad C \in \mathbb{R}.$$

Then

$$t = \frac{1}{e^{x} - C}$$

Therefore the general solution of (6) is

$$y = t + e^{x} \implies y = \frac{1}{e^{x} - C} + e^{x}.$$

2^{nd} order differential equations

Definition 8

A second order differential equation is an relation in the form

$$F(x, y, y', y'') = 0, \tag{7}$$

between the variable x, the function $y(x)$ and these two first derivatives.

Example 9

- $y'' + \omega^2 y = 0$, admits for solutions on $\mathbb{R}$ the functions

$$\varphi_1(x) = \sin \omega x,\ \ \varphi_2(x) = \cos \omega x$$

- $y'' = 0$, admits for solutions all polynomial of the form $ax + b$, with a and b two arbitrary constants.

Cancel y

Let

$$F(x, y', y'') = 0.$$

We make $y' = z$, then the equation become

$$F(x, z, z') = 0.$$

Example 10

$$y'' + y'^2 = 0, \quad \text{we set } y' = z, \text{ then } z' + z^2 = 0 \Rightarrow -\frac{dz}{z^2} = dx$$

$$\Rightarrow z = \frac{1}{x+c} \Rightarrow dy = \frac{dx}{x+C}$$

$$y = \ln(x+C) + K, \quad C,\ K \text{ are constants}$$

Cancel y

Exercise

Integrate the equation $xy'' + 2y' = 0$.

Linear equations

Definition 11

A second order linear differential equation is an equation of the form

$$a(x)y'' + b(x)y' + c(x)y = f(x),$$

où $a(x),\ b(x),\ c(x)$ and $f(x)$ are functions.

We associate with this equation the equation without a second member called **homogeneous equation**, i.e.

$$a(x)y'' + b(x)y' + c(x)y = 0,$$

with right hand side called **complete equation.**

Second order linear differential equations

Theorem 12

The general solution is obtained by adding to a particular solution of the complete and equation the general solution of the homogeneous equation (i.e. without a second member)

$$y_G = y_P + y_H.$$

✓ Solving the equation without a second member:
If we know $y_1,\ y_2$ a particular solutions, then the general solution is written as the form $y = \lambda_1 y_1 + \lambda_2 y_2$ and the two solutions must be linearly independent, which resulting in

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0. \quad \text{Is called the Wronskian}$$

Second ODE

Remark 1

There is not a general method we flow us to find the general solution in finite form for the second order differential equations of coefficients variables. However, such a method exists for equations with constants coefficients which will be the subject of the following paragraph.

Linear differential equations with constant coefficients

They are of the form

$$ay'' + by' + cy = f(x), \tag{8}$$

where $a,\ b,\ c$ are real constants and $f(x)$ is a function.

Solving this type of equations:

$\star$ **We search** y_H of (9), i.e $f(x) = 0.$ in the form $y = e^{rx}$ and we replace in

$$ay'' + by' + cy = 0, \tag{9}$$

we obtain

$$e^{rx}(ar^2 + br + c) = 0 \Rightarrow \boxed{ar^2 + br + c = 0},$$

called the **characteristic equation**

Homogeneous solution (y_H): We distinguish three cases

- If $\Delta = b^2 - 4ac > 0$, then

$$r_1 = \frac{-b - \sqrt{\Delta}}{2a}, \; r_2 = \frac{-b + \sqrt{\Delta}}{2a}$$

two distinct real roots. The general solution of the homogeneous equation (H.E) is given by the formula

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}, \quad C_1, \; C_2 \quad \text{are real constants.}$$

- If $\Delta = b^2 - 4ac = 0$, the characteristic equation has a double real root $r_1 = r_2 = r = \dfrac{-b}{2a}$. Then the general solution of (H.E) is given by

$$y = (C_1 x + C_2)e^{rx}.$$

- If $\Delta = b^2 - 4ac < 0$, in this case the characteristic equation has two conjugated complex roots $r_1 = \alpha + i\beta, \quad r_2 = \alpha - i\beta$. then the general solution of (H.E) is given by the following formula

$$y = e^{\alpha x}(c_1 \cos(\beta x) + C_2 \sin(\beta x)) \quad C_1,\ C_2 \quad \text{are real constants.}$$

Example

To solve the equation

$$y'' + y' + y = 0,$$

the associate characteristic equation is

$$r^2 + r + 1 = 0 \implies \Delta = -3 < 0,$$

we have two complex roots conjugate

$$r_1 = \frac{-1}{2} + i\frac{\sqrt{3}}{2}, \qquad r_2 = \frac{-1}{2} - i\frac{\sqrt{3}}{2}$$

The general solution is given by

$$y_H = e^{\frac{-1}{2}x}\left(C_1 \cos\frac{\sqrt{3}}{2}x + C_2 \sin\frac{\sqrt{3}}{2}x\right), \quad C_1,\ C_2 \in \mathbb{R}.$$

Particular solution (y_P)

$\star$ **we search the general solution of complete equation** (8) i.e. $f(x) \neq 0$.

$\circ$ Case where $f(x) \in \mathbb{R}_n[X]$

- $c \neq 0$. we replace y by a polynomial of degree n, then we identify.
- $c = 0,\ b \neq 0$ we replace y by a polynomial of degree $n+1$, then we identify.
- $c = 0,\ b = 0$ and $a \neq 0$ we integrate twice the function $f(x)$.

Example: y_H

$$y'' + 3y' + 2y = 2x^2 + 8x + 7,$$

The characteristic equation is:

$$r^2 + 3r + 2 = 0 \implies r_1 = -1, \quad r_2 = -2$$

The homogeneous solution is

$$y_H = C_1 e^{-x} + C_2 e^{-2x}$$

Example: y_P y_G

y_P in the form $y_P = ax^2 + bx + c$

$$y'_P = 2ax + b \implies y''_P = 2a$$

By replacing in complete equation and after identification, we obtain

$$a = 1, \quad b = \frac{5}{2}, \quad c = \frac{-5}{4}$$

Then $y_P = x^2 + \frac{5}{2}x + \frac{-5}{4}$ and

$$y_G = C_1 e^{-x} + C_2 e^{-2x} + x^2 + \frac{5}{2}x + \frac{-5}{4}.$$

Particular solution (y_P)

- Case where $f(x) = e^{\alpha x} P_n(x)$,

We search a particular solution, we make $y = e^{\alpha x} P(x)$, (P arbitrary polynomial.) The equation (8) can be written

$$aP'' + (2\alpha a + b)P' + (a\alpha^2 + b\alpha + c)P = P_n(x).$$

We have in the first case.

Particular solution (y_P)

○ Case where $f(x) = A\cos(\alpha x) + B\sin(\alpha x), \quad \alpha,\ A,\ B \in \mathbb{R}$.

• If $i\alpha$ is not solution of characteristic equation, we search y_P in the form $y_P = A'\cos(\alpha x) + B'\sin(\alpha x)$

• If $i\alpha$ is a root of a characteristic equation (necessarily simple), we search y_P in the form $y_P = x(A'\cos(\alpha x) + B'\sin(\alpha x))$

General Method

In the general case. To solve the complete equation, we apply the variation of constants method as follow:
Let $y = C_1 y_1 + C_2 y_2$ general solution of the equation without a second member. Search for a particular solution by the variation of constants method we make

$$y = C_1(x)y_1(x) + C_2(x)y_2(x) \quad \text{with } C_1'(x)y_1(x) + C_2'(x)y_2(x) = 0.$$

After calculation, we will have two equations with two unknowns

$$\begin{cases} C_1'(x)y_1'(x) + C_2'(x)y_2'(x) = f(x) \\ C_1'(x)y_1(x) + C_2'(x)y_2(x) = 0 \end{cases}$$

This system has one and only one solution because $y_1,\ y_2$ are linearly independent, therefore $C_1(x),\ C_2(x)$ exists.

Example

Example 13

Solve the second order differential equation $y'' + y = x \sin x$.

Solution

Solve the equation without second member

$$y'' + y = 0$$

Characteristic equation :

$$r^2 + 1 = 0 \Rightarrow r = \pm i,$$

so

$$y_H = C_1 \cos x + C_2 \sin x.$$

Solution

with second member:
we use the variation of constants method.

$$\begin{cases} C_1'(x)y_1(x) + C_2'(x)y_2(x) = 0 \\ C_1'(x)y_1'(x) + C_2'(x)y_2'(x) = x\sin x \end{cases}$$

$$\begin{cases} C_1'\cos x + C_2'\sin x = 0 \\ -C_1'\sin x + C_2'\cos x = x\sin x \end{cases}$$

Solution

$$\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1.$$

Then $C_1' = -x\sin^2 x, \quad C_2' = x\cos x\sin x$
After integration we have:
$C_1 = -\frac{x^2}{4} + \frac{x}{4}\sin 2x + \frac{1}{8}\cos 2x, \quad C_2 = \frac{x}{2}\cos 2x - \frac{1}{4}x + \frac{1}{8}\sin 2x$
We replace C_1, C_2 in y_H we obtain the general solution y_G.

Exercise

$$y'' - y = e^{2x} - e^{x} ....;(2)$$

Characteristic equation is

$$r^2 - 1 = 0, \quad r = \pm 1$$

Solution without second member is

$$C_1 e^{-x} + C_2 e^{x}$$

then we find y_P, use **the variation constant method** as follow: we put $y_1 = e^{-x}, \quad y_1 = e^{x}$ and solve the system equations

$$\begin{cases} C_1' y_1 + C_2' y_2 = 0 \\ C_1' y_1' + C_2' y_2' = f(x) \end{cases}$$

$$\begin{cases} C_1' e^{-x} + C_2' e^{x} = 0......(1) \\ -C_1' e^{-x} + C_2' e^{x} = f(x)...(2) \end{cases}$$

$$(1) + (2) : C_2' = \frac{1}{2}(e^{x} - 1) \Rightarrow C_2 = \frac{1}{2}e^{x} - \frac{x}{2} + K_2$$

$$(2) - (1) : C_1' = -\frac{1}{2}(e^{3x} - e^{2x}) \Rightarrow C_1 = -\frac{1}{6}e^{3x} + \frac{1}{4}e^{2x} + K_1$$

then

$$y_G = K_1 e^{-x} + K_2 e^{x} + \frac{1}{3}e^{2x} + \frac{1}{4}(1 - 2x)e^{x}$$

End Chapter 3