Simple Integrals

Presented by : Dr : M.TOUAHRIA

Mohamed Boudiaf University of Msila
Faculty of Mathematics and Computer Science
Computer Science Department
Level : First year bachelor's degree

9 mars 2026

Motivation

We will introduce the integral using an example. We consider the exponential function

$$f : x \mapsto e^x.$$

We want to calculate the area A below the graph of f and between the vertical lines of equations $x = 0,\ x = 1,$ and x-axis.
We approximate this area by sums of areas of the rectangles located under the graph. More precisely, let $n \geq 1$ an integer, let's divide our interval $[0, 1]$ using the subdivision

$$(0, \frac{1}{n}, \frac{2}{n}, ..., \frac{i}{n}, ..., \frac{n-1}{n}, 1).$$

each interval has $\left[\dfrac{i-1}{n}, \dfrac{i}{n}\right]$ as a basis and $f(\frac{i-1}{n}) = e^{\frac{i-1}{n}}, \quad i = 1, ..., n$ as height, then the area of each rectangle equal

$$(\frac{i}{n} - \frac{i-1}{n}).e^{\frac{i-1}{n}} = \frac{1}{n}.e^{\frac{i-1}{n}}$$

The sum of the areas is then calculated as the sum of a geometric sequence : We consider the "lower rectangles" A_{inf}

$$\sum_{n=1}^{n} \frac{e^{\frac{i-1}{n}}}{n} = \frac{1}{n}\sum_{n=1}^{n}(e^{\frac{1}{n}})^{i-1} = \frac{1}{n}.\frac{1-(e^{\frac{1}{n}})^n}{1-e^{\frac{1}{n}}}$$

$$= \frac{\frac{1}{n}}{e^{\frac{1}{n}}-1}(e-1) \longrightarrow (e-1) \text{ when } n \to +\infty$$

Draboux sum

Let $I = [a, b]$ be an interval of $\mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$ be a bounded function. We set

$$m = \inf_{x \in [a,b]} f(x) \quad \text{and} \quad M = \sup_{x \in [a,b]} f(x)$$

Definition 1

An ordered subdivision of the segment $I = [a, b]$ is any strictly increasing sequence $(x_i)_{i=0}^{n}$ such that

$$a = x_0 < x_1 < x_2 < ...x_{n-1} < x_n = b.$$

In this case, we write : $d = \{x_0, x_1, ..., x_n\}$.

For an example, let $[a, b] = [0.2]$ and $d = \{0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1, \frac{3}{2}, 2\}$.

Draboux sum

Definition 2

For $0 \leq i < n-1$, we set
$m_i = \inf_{x\in[x_i,x_{i+1}]} f(x)$ and $M_i = \sup_{x\in[x_i,x_{i+1}]} f(x)$ and

$$s(f,d) = \sum_{i=0}^{n-1} m_i(x_{i+1}-x_i), \quad S(f,d) = \sum_{i=0}^{n-1} M_i(x_{i+1}-x_i).$$

The quantities s and S are called the Draboux sums.

Example 3

Let $f(x) = \dfrac{1}{x}$ and $[a,b] = [1,2]$. Let $d = \{1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 1, 2\}$.

$$s(f,d) = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = 0.63, \quad S(f,d) = \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} = 0.75$$

$$\boxed{0.63 < \ln 2 < 0.7}$$

Integable function

Let f be a function defined on a closed interval $[a, b]$. we reply the same construction as previously. What will replace the rectangles will be *step functions*. If the limit of the areas below equals the limit of the areas above we call this common limit **the integral** of f that we notes

$$\int_a^b f(x)dx$$

Remark 1

It is not always true that these limits (that's way, the limit of area bellow and the limit of area above) are equal, the integral is only defined for integrable functions. Fortunately we will see that if the function f is continuous then it is integrable.

Properties of simple integral

Let $f,\ g$ continuous functions on $[a,b]$ and $\lambda, \mu \in \mathbb{R}$

1. Linearity : The integral is a linear form on the vector space of piecewise continuous functions on a segment $[a,b]$. i.e.

$$\int_a^b (\lambda f + \mu g)(x)dx = \lambda \int_a^b f(x)dx + \mu \int_a^b g(x)dx.$$

This equivalent to

$$\begin{cases} \int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx \\ \int_a^b \lambda f(x)dx = \lambda \int_a^b f(x)dx. \end{cases}$$

2. Equality except at a finite number of points
If $f,\ g$ are two piecewise continuous functions that are equal except at a finite number of points then

$$\int_a^b f(x)dx = \int_a^b g(x)dx$$

1. $$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

2. Chasles relation : : Additivity of the integral :

$$\forall c \in [a,b] : \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

3. Positivity :

$$(\forall x \in [a,b], f(x) \geq 0) \Rightarrow (\int_a^b f(x)dx) \geq 0.$$

Moreover if

$$(f \text{continuous on} [a,b], f \neq 0; \forall x \in [a,b], f(x) \geq 0) \Rightarrow (\int_a^b f(x)dx) > 0.$$

Properties of simple integral

1. Monotonicity : Let f, g two functions on $[a, b]$ is integrable. If

$$\forall x \in [a, b], f(x) \leq g(x) \text{ alors } \int_a^b f(x)dx \leq \int_a^b g(x)dx$$

2. **Cauchy Schwarz inequality** : si f and g are continuous on $[a, b]$ we have the inequality

$$\left(\int_a^b f(x)g(x)dx\right)^2 \leq \int_a^b f^2(x)dx \int_a^b g^2(x)dx$$

3. Mean value inequalities
Let $f,\ g$ be piecewise continuous functions on $[a, b]$. Then

$$\left|\int_a^b f(x)g(x)dx\right| \leq \sup_{[a,b]} |f| \int_a^b |g(x)|dx$$

In particular, we have

title

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx \leq (b-a)\sup_{[a,b]} |f|.$$

The quantity

$$\frac{\int_a^b f(x)dx}{b-a},$$

is the mean value of f on $[a, b]$.

Theorem 4

Second mean value theorem.
Let f, g be continuous functions such that $\forall x \in [a,b] : g(x) \geq 0$. Then

$$\exists c \in [a,b] : \int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx.$$

Proof :
For every $x \in [a,b]$, we have

$$m \leq f(x) \leq M \iff mg(x) \leq f(x)g(x) \leq Mg(x)$$

$$m\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq M\int_a^b g(x)dx$$

$$m \leq \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \leq M$$

By the intermediate value theorem, there exists $c \in [a, b]$ such that

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b f(x)dx.$$

Remark 2

1. *If g is negative over $[a, b]$, we take $-g$. Thus the theorem remains true if g does not change sign.*
2. *If g changes sign, the theorem is false in general, as shown by the following example.*

Example 5

$$f(x) = g(x) = x - 1, \text{ on the interval}[0, 2], \quad \int_0^2 f(x)g(x)dx = \frac{2}{3}.$$

But

$$f(c)\int_0^2 g(x)dx = 0. \quad \forall x \in [0, 2]. \quad \color{red}{CHEK}$$

Definition 6

A function F is a primitive of a function f on an interval if F is differentiable on the interval and satisfies the equation, $F'(x) = f(x)$, or, is the same of,

$$dF(x) = f(x)dx \iff F(x) = \int f(x)dx.$$

Remark 3

- *The difference between two primitives of* f *is a constant.*
- *If* x_0 *is an element of* I *there exists a unique primitive of* f *zero at* x_0 :

$$F(x) = \int_{x_0}^{x} f(t)dt$$

Remark 3

For example :

$$\ln x = \int_1^x \frac{1}{t} dt$$

- The compute of an integral then comes down to the compute of the primitives.
- If f is a continuous function on interval, we note $\int f(x)dx$ is a primitive of f on the interval.

To find an a primitive of a function f one may be lucky enough to recognize that f is the derivative of a well-known function. This is unfortunately very rarely case, and we do not know the primitives of most functions. However, we will see two techniques which allow us to calculate integrals and primitives : integration by parts and by change of variable.

Table of Usual Functions and Their Primitives

Function	Primitive
$x^{\alpha}\ (\alpha \neq -1)$	$\int x^{\alpha}dx = \frac{x^{\alpha+1}}{\alpha+1} + C$
$\frac{1}{x}$	$\int \frac{1}{x}dx = \ln\lvert x\rvert + C$
$\sin x$	$\int \sin x dx = -\cos x + C$
$\cos x$	$\int \cos x = \sin x + C$
$\tan x$	$\int \tan x dx = -\ln\lvert\cos x\rvert + C$
$\frac{1}{\cos^2 x}$	$\int \frac{1}{\cos^2 x}dx = \tan x + C$
$\frac{1}{\sin^2 x}$	$\int \frac{1}{\sin^2 x}dx = -\cot x + C$
$\cos(ax+b) \quad a \neq 0$	$\int \cos(ax+b)\frac{1}{a}\sin(ax+b) + C$
$\sin(ax+b) \quad a \neq 0$	$\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + C$
$(x-a)^{\alpha}\ (\alpha \neq -1)$	$\int (x-a)^{\alpha}dx = \frac{(x-a)^{\alpha+1}}{\alpha+1} + C$

Table of Usual Functions and Their Primitives

$\dfrac{1}{x-a}$	$\int \dfrac{1}{x-a}dx = \ln\lvert x-a\rvert + C$
e^x	$\int e^x dx = e^x + C$
$e^{ax} \quad a \neq 0$	$\dfrac{1}{a}e^{ax} + C$
$\dfrac{1}{\sqrt{x}}$	$\int \dfrac{1}{\sqrt{x}}dx = 2\sqrt{x} + C$
$\dfrac{1}{1+x^2}$	$\int \dfrac{1}{1+x^2} = \arctan x + C$
$\dfrac{1}{\sqrt{1-x^2}}$	$\int \dfrac{1}{\sqrt{1-x^2}} = \arcsin x + C$
$-\dfrac{1}{\sqrt{1-x^2}}$	$\int -\dfrac{1}{\sqrt{1-x^2}}dx = \arccos x + C$
$\sinh x = Shx$	$\int Shxdx = \cosh x + C = Chx + C$
$\cosh x = Chx$	$\int Chxdx = \sinh x + C = Shx + C$

Table of Usual Functions and Their Primitives

$\dfrac{1}{\sqrt{x^2-1}}$	$\int \dfrac{1}{\sqrt{x^2-1}}dx = Argchx = \ln\left\vert x+\sqrt{x^2-1}\right\vert + C$
$\dfrac{1}{\sqrt{x^2+1}}$	$\int \dfrac{1}{\sqrt{x^2+1}}dx = Argshx = \ln\left\vert x+\sqrt{x^2+1}\right\vert + C$
$\dfrac{1}{1-x^2}$	$\int \dfrac{1}{1-x^2}dx = Argthx = \dfrac{1}{2}\ln\left\vert\dfrac{1+x}{1-x}\right\vert$

Techniques to calculate integral

1. By change variable

Theorem 7

Let u a function defined on J and $v : I \to J$ is bijection of class $\mathcal{C}^1$. For all $a, b \in J$

$$\int_a^b u(v(t))v'(t)dt = \int_{v(a)}^{v(b)} u(x)dx$$

Proof : u is continuous, we consider U the primitive function of u over J. The function $U \circ v$ is differentiable, as component of two differentiable functions and we have : $(U \circ v)' = u \circ v.v'$. therefore

$$\int_a^b u(v(t))v'(t)dt = \int_a^b (U \circ v)'(t)dt = U \circ v(b) - U \circ v(a)$$

$$U(v(b)) - U(v(a)) = \int_{v(a)}^{v(b)} u(x)dx.$$

This completes the proof.

Example

Example 8

Compute the following integrals :

$$\int \frac{x}{\sqrt{1+x^2}}dx, \quad \int_0^{\frac{\pi}{2}} x\cos(x^2)dx. \quad \int \frac{x}{1+\sqrt{1+x}}dx$$

$$\int \frac{x}{\sqrt{1+x^4}}dx, \quad \int \cos^3 x dx.$$

Solution

Let $t = 1 + x^2 \implies dt = 2dx$

$$\int \frac{x}{\sqrt{1+x^2}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{t}} = \sqrt{t} = \sqrt{1+x^2} + C.$$

Let $t = x^2 \implies dt = 2xdx$

$$\int_0^{\frac{\pi}{2}} x\cos(x^2)dx = \frac{1}{2}\int_0^{\frac{\pi^2}{4}} \cos(t)dt = \frac{1}{2}[\sin t]_0^{\frac{\pi^2}{4}} = \boxed{\frac{1}{2}\sin(\frac{\pi^2}{4})}.$$

Let $t = \sqrt{1+x} \implies dt = \frac{dx}{2\sqrt{1+x}} \implies 2tdt = dx$

$$\int \frac{x}{1+\sqrt{1+x}}dx = 2\int (t^2 - t) = 2\frac{t^3}{3} - t^2 + C = \boxed{\frac{2}{3}(1+x)^{\frac{3}{2}} - (1+x) + C}.$$

Solution

Let $t = x^2 \implies dt = 2xdx$

$$\int \frac{x}{\sqrt{1+x^4}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{1+t^2}} = \frac{1}{2}argsh(t) = \frac{1}{2}\ln|t+\sqrt{1+t^2}| + C$$

$$\frac{1}{2}\ln|x^2+\sqrt{1+x^4}| + C$$

We have $\cos^3 x = \cos x(1-\sin^2 x)$, then let $t = \sin x \implies dt = \cos x dx$

$$\int \cos^3 x dx = \int (1-t^2)dt = t - \frac{t^3}{3} = \sin x - \frac{\sin^3 x}{3} + C.$$

Techniques to calculate integral

1. By Parts

Theorem 9

Lets u and v two functions of the class $\mathcal{C}^1$ over the interval $[a,b]$.

$$\int_a^b u(x)v'(x)dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)dx \iff \int_a^b udv = [uv]_a^b - \int_a^b vdu$$

We have

$$(u(x)v(x))' = u'(x)v(x) + u(x)v'(x)$$

By Integration, we obtain

$$\int_a^b (u(x)v(x))'dx = \int_a^b u'(x)v(x)dx + \int_a^b u(x)v'(x)dx \quad \text{then}$$

$$[u(x)v(x)]_a^b = \int_a^b u'(x)v(x)dx + \int_a^b u(x)v'(x)dx.$$

Example 10

Give the primitive functions of :

$$\int \arcsin x dx, \quad \int \ln x dx, \quad \int \arctan x dx, \quad \int_1^2 x^2 e^x dx$$

Solution

- $$\int \arcsin x dx = [x \arcsin x] - \int \frac{x dx}{\sqrt{1-x^2}} = x \arcsin x + \sqrt{1-x^2} + C$$
- $$\int \ln x = x \ln x - \int x(\frac{1}{x}) dx = x \ln x - x + C.$$
- $$\int \arctan x dx = [x \arctan x] - \int \frac{x}{1+x^2} dx = x \arctan x - \frac{1}{2} \ln(1+x^2) + C.$$

Real polynomial

Definition 11

Any function

$$\begin{cases} P : \mathbb{R} \to \mathbb{R} \\ x \longmapsto P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0, \end{cases}$$

where $(a_0, a_1, ..., a_n) \in \mathbb{R}^n$ and $n \in \mathbb{N}$, is called a **real polynomial**. The numbers $a_0, a_1, ..., a_n$ are called the **coefficients** of the polynomial. If $a_n \neq 0$, the polynomial is said to be of degree n and we write $\deg P = n$.

We denote by $\mathbb{R}[X]$ respectively $\mathbb{R}_n[X]$ the set of real polynomials (respectively of degree equal to) n.

Theorem 12

Let P and Q be two real polynomials. Suppose that $Q \neq 0$. Then

$$\exists! R, S \in \mathbb{R}_n[X] : P = SQ + R \quad \textit{with } \deg R < \deg Q.$$

R and S are called respectively the remainder and the quotient of the division of P by Q

Proof : Uniqueness of R and S.
Suppose that

$$\begin{cases} P = QS + R, & \deg R < \deg Q \\ P = QS' + R', & \deg R' < \deg Q' \end{cases}$$

By subtraction, we obtain

$$Q(S - S') + R - R' = 0,$$

first, we have $\deg(R - R') \leq \max(\deg R, \deg R') < \deg Q$
second, $\deg Q \leq \deg Q + \deg(S - S') = \deg(R - R') < \deg Q$ contradiction.

The existence of R and S.

- If $\deg P < \deg Q$. Take $S = 0$ and $P = R$.
- If $\deg P \geq \deg Q$. Divide P by Q

Example 13

Let

$$P(x) = 3x^7 + x^6 - 5x^5 + x^3 - 2x - 1,$$

$$Q(x) = x^4 - 3x^3 + x^2 + x - 3. \ \deg P = 7, \ \deg Q = 4$$

By euclidean division we obtain CHEK !

$$S(x) = 3x^3 + 10x^2 + 22x + 53$$

$$R(x) = 1373x^3 - 45x^2 + 11x + 158. \ \deg S = 3, \ \deg R = 3 < \deg Q$$

Definition 14

Let P and Q be two real polynomials. Such that $Q \neq 0$. The function

$$x \mapsto \frac{P(x)}{Q(x)},$$

is called a rational fraction (or rational function)

We need decompose this fraction into simple elements of the first species in the form

$$x \mapsto \frac{A}{(x-a)^n}$$

and of the second species in the form

$$x \mapsto \frac{Ax+B}{[(x-a)^2+b^2]^n}$$

where $A, B, a, b \in \mathbb{R}$ and $n \in \mathbb{N}$.

Decomposition of simple fractions

However, before this decomposition the following rules must be verified

1. If $\deg(P(x)) \geq \deg(Q(x))$ we must do the euclidean division

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)} \quad \text{with} \deg(R(x)) < \deg(Q(x))$$

2. Decompose the fraction $\dfrac{R(x)}{Q(x)}$ i.e we put $Q(x)$ as a product of factors
   - $1-st$ degree factor not repeated.
   - $1-st$ degree factor repeated n times.
   - $2-nd$ degree factor not repeated.
   - $2-nd$ degree factor repeated n times.

As follow :

$$Q(x) = (x-x_1)^{n_1}(x-x_2)^{n_2}...(x-x_j)^{n_j}$$

$$[(x-a_1)^2+b_1^2]^{m_1}[(x-a_2)^2+b_2^2]^{m_2}...[(x-a_k)^2+b^2]^{m_k}$$

Decomposition of simple fractions

$$\frac{R(x)}{Q(x)} = \frac{A_{11}}{(x-x_1)} + \frac{A_{12}}{(x-x_1)^2} + \ldots + \frac{A_{1n_1}}{(x-x_1)^{n_1}}$$

$$+\frac{A_{21}}{(x-x_2)} + \frac{A_{22}}{(x-x_2)^2} + \ldots + \frac{A_{2n_2}}{(x-x_2)^{n_2}}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\frac{A_{j1}}{(x-x_j)} + \frac{A_{j2}}{(x-x_j)^2} + \ldots + \frac{A_{jn_j}}{(x-x_j)^{n_j}}$$

Decomposition of simple fractions

$$
\begin{aligned}
&+\frac{A'_{11}x+B_{11}}{[(x-a_1)^2+b_1^2]}+\frac{A'_{12}x+B_{12}}{[(x-a_1)^2+b_1^2]^2}+\dots+\frac{A'_{1m_1}x+B_{1m_1}}{[(x-a_1)^2+b_1^2]^{m_1}}\\
&+\frac{A'_{21}x+B_{21}}{[(x-a_2)^2+b_2^2]}+\frac{A'_{22}x+B_{22}}{[(x-a_2)^2+b_2^2]^2}+\dots+\frac{A'_{2m_2}x+B_{2m_2}}{[(x-a_2)^2+b_2^2]^{m_2}}\\
&\qquad\qquad\qquad\qquad\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&+\frac{A'_{1k}x+B_{1k}}{[(x-a_k)^2+b_k^2]}+\frac{A'_{2k}x+B_{2k}}{[(x-a_k)^2+b_k^2]^2}+\dots+\frac{A'_{km_k}x+B_{km_k}}{[(x-a_k)^2+b_2^2]^{m_k}}
\end{aligned}
$$

Example 15

Decompose the following fraction into simple fractions :

$$\frac{x+1}{(x-1)(x-2)^2(x^2+1)}, \quad \frac{1}{x^4+1}, \quad \frac{x^3}{x^3+x^2+x+1}.$$

From the fraction,

$$\frac{x+1}{(x-1)(x-2)^2(x^2+1)},$$

we have $\deg(x-1) < \deg((x-1)(x-2)^2(x^2+1))$. Hence

- $x-1, \quad 1-st$ degree no repeat.
- $(x-2)^2, \quad 1-st$ degree factor repeated 2 times.
- $x^2+1, \quad 2-nd$ degree factor not repeated.

Decomposition of simple fractions

$$\frac{x+1}{(x-1)(x-2)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{Dx+E}{x^2+1}$$

$$x+1 = A(x^4-4x^3+5x^2-4x+4) + B(x^4-3x^3+3x^2-3x+2)$$
$$+C(x^3-x^2+x-1) + (Dx+E)(x^3-5x^2+8x-4)$$

Now collect coefficients :

$$x+1 = (A+B+D)x^4 + (-4A-3B+C-5D+E)x^3$$

$$+(5A+3B-C+8D-5E)x^2+(-4A-3B+C-4D+8E)x+(4A+2B-C-4E)$$

Example

Since the left side is $x+1$, we obtain the system :

$$\begin{cases} A+B+D=0 & (1) \\ -4A-3B+C-5D+E=0 & (2) \\ 5A+3B-C+8D-5E=0 & (3) \\ -4A-3B+C-4D+8E=1 & (4) \\ 4A+2B-C-4E=1 & (5) \end{cases}$$

Adding the equations, we find $A=1$.
In general after calculating we obtain

$$\boxed{A=1, \quad B=-\frac{22}{25}, \quad C=\frac{15}{25}, \quad D=-\frac{3}{25}, \quad E=-\frac{4}{25}.}$$

Decomposition of simple fractions

A rational fraction is written in a unique way as the sum of a finite number of simple fractions and a polynomial. So, calculating the primitive of a rational fraction amounts to calculating the primitive of simple fractions.

$$\frac{x+1}{(x-1)(x-2)^2(x^2+1)} = \frac{1}{x-1} - \frac{22}{25}\frac{1}{x-2} + \frac{15}{25}\frac{1}{(x-2)^2} - \frac{1}{25}\frac{3x+4}{x^2+1}$$

Therefore the primitive is

$$\int \frac{x+1}{(x-1)(x-2)^2(x^2+1)}dx = \int \frac{1}{x-1}dx - \frac{22}{25}\int \frac{1}{x-2}dx$$

$$+\frac{15}{25}\int \frac{1}{(x-2)^2}dx - \frac{1}{25}\int \frac{3x+4}{x^2+1}dx$$

Primitive of fraction function

1. 1-st type

$$\int \frac{1}{(x-a)^n}dx = \begin{cases} \ln|x-a| & \text{si } n=1 \\ \dfrac{-1}{(n-1)(x-a)^{n-1}} & \text{si } n>1 \end{cases}$$

2. 2-nd type

$$\int \frac{Ax+B}{\left[(x-a)^2+b^2\right]^n}\,dx$$

We write :

$$Ax+B = A(x-a)+aA+B$$

Hence,

$$= \int \frac{A(x-a)+aA+B}{\left[(x-a)^2+b^2\right]^n}dx$$

Primitive of fraction function

$$= \frac{A}{2} \int \frac{2(x-a)}{\left[(x-a)^2 + b^2\right]^n} dx + (aA + B) \int \frac{1}{\left[(x-a)^2 + b^2\right]^n} dx$$

For the first integral, clear

$$\frac{A}{2} \int \frac{2(x-a)}{\left[(x-a)^2 + b^2\right]^n} dx = \frac{-\frac{A}{2}}{(n-1)\left[(x-a)^2 + b^2\right]^{n-1}} + C_1.$$

For the second integral, we are led to compute

$$I_n = \int \frac{1}{\left[(x-a)^2 + b^2\right]^n} dx$$

Set :

$$I_n = \frac{1}{b^{2n-1}} \int \frac{1}{\left(\left(\frac{x-a}{b}\right)^2 + 1\right)^n} dx$$

Primitive of fraction function

We make the change variable $\frac{x-a}{b} = t \implies dx = bdt$

$$I_n = \frac{1}{b^{2n-1}} \int \frac{dt}{(t^2+1)^n} = \frac{1}{b^{2n-1}} J_n$$

For $n = 1$:

$$J_1 = \int \frac{dt}{t^2+1} = \arctan t + C_1$$

Thus,

$$I_1 = \frac{1}{b} \arctan \frac{x-a}{b} + C_1.$$

Now compute J_{n-1} :

Primitive of fraction function

$$J_{n-1} = \int \frac{1}{(t^2+1)^{n-1}} dt$$

Using integration by parts :
Let

$$du = 1 \implies u = t, \qquad v = \frac{1}{(t^2+1)^{n-1}} \implies dv = \frac{-2(n-1)t}{(t^2+1)^n} dt$$

Then

$$J_{n-1} = \frac{t}{(t^2+1)^{n-1}} + 2(n-1) \int \frac{t^2}{(t^2+1)^n} dt \quad \text{because } t^2 = (t^2+1) - 1,$$

we have

$$J_{n-1} = \frac{t}{(t^2+1)^{n-1}} + 2(n-1) \left(\int \frac{t^2+1}{(t^2+1)^n} dt - \int \frac{1}{(t^2+1)^n} dt \right)$$

$$= \frac{t}{(t^2+1)^{n-1}} + 2(n-1)J_{n-1} - 2(n-1)J_n$$

Which implies :

$$J_n = \frac{2n-3}{2n-2}J_{n-1} + \frac{1}{2n-2}\frac{t}{(t^2+1)^{n-1}}$$

Therefore,

$$J_n = \frac{2n-3}{2n-2}J_{n-1} + \frac{b^{2n-1}}{2n-2}\frac{x-a}{\left[(x-a)^2+b^2\right]^{n-1}}$$

Example 16

Give the primitives functions of

$$\int \frac{x^5 - x^2 + 2x + 1}{x^3 - 1} dx, \quad I_2 = \int \frac{1}{[x^2 + 2x + 2]^2} dx.$$

Example

Since the degree of the numerator is greater than the degree of the denominator, we first do euclidean division.

$$\frac{x^5 - x^2 + 2x + 1}{x^3 - 1} = x^2 + \frac{2x + 1}{x^3 - 1}. \qquad \color{red}{CHEK}$$

Thus,

$$\int \frac{x^5 - x^2 + 2x + 1}{x^3 - 1}\,dx = \int x^2\,dx + \int \frac{2x + 1}{x^3 - 1}\,dx.$$

Example

We decompose into partial fractions :

$$\frac{2x+1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}.$$

After identification, we obtain :or

- Multiplying the previous equation by $(x-1)$ and then setting $x=1$, the result is $\boxed{A=1}$.
- Multiplying the same equation by x and take $x \to +\infty$, the result is $A+B=0 \implies \boxed{B=-A=-1}$
- For $x=0$ the previous equation become $-1=-A+C \implies \boxed{C=0}$

Hence,

$$\frac{2x+1}{x^3-1} = \frac{1}{x-1} + \frac{-x}{x^2+x+1}.$$

Example

We now integrate term by term :

$$\int x^2 dx = \frac{x^3}{3} + C_1,$$

$$\int \frac{1}{x-1} dx = \ln|x-1| + C_2.$$

For the last term, write :

$$\int \frac{-x}{x^2+x+1} dx = -\frac{1}{2} \int \frac{2x+1}{x^2+x+1} dx + \frac{1}{2} \int \frac{dx}{x^2+x+1}.$$

We obtain :

$$-\frac{1}{2} \ln(x^2+x+1) + \frac{1}{2} \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}.$$

Example

Using the standard formula

$$\int \frac{dx}{(x-a)^2+b^2} = \frac{1}{b}\int \frac{\frac{1}{b}dx}{(\frac{x-a}{b})^2+1} = \frac{1}{b}\arctan\left(\frac{x-a}{b}\right),$$

with

$$a = -\frac{1}{2}, \qquad b = \frac{\sqrt{3}}{2},$$

we get

Example

$$\frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{3}\arctan\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)$$

$$= \sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+C_3.$$

Finally

$$\int \frac{x^5-x^2+2x+1}{x^3-1}\,dx = \frac{x^3}{3}+\ln|x-1|-\ln(x^2+x+1)+\sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+C,$$

such that $C = C_1+C_2+C_3$.

Example

$$I_2 = \int \frac{1}{[x^2+2x+2]^2}\,dx.$$

Let $t = x+1$. Then

$$I_2 = \int \frac{1}{[(x+1)^2+1]^2}\,dx = \int \frac{1}{(t^2+1)^2}\,dt.$$

Write

$$\frac{1}{(t^2+1)^2} = \frac{1}{t^2+1} - \frac{t^2}{(t^2+1)^2}.$$

After integration, we obtain

$$\int \frac{1}{[x^2+2x+2]^2}\,dx = \frac{1}{2}\arctan(x+1) + \frac{x+1}{2(x^2+2x+2)} + C.$$

Exercise

Give the primitive functions of

$$\int \frac{x^3+x^2+2}{(x^2+2)^2}dx; \quad \int \frac{x^4}{x^2-1}\,dx, \quad \int \frac{x^6+x+1}{x^5-x^4+2x^3-2x^2+x+1}dx.$$

We have $deg(x^4) > \deg(x^2-1)$ by euclidean division we obtain :

$$\frac{x^4}{x^2-1} = x^2+1+\frac{1}{x^2-1}.$$

Solution

By integrate term by term we find :

$$\int \frac{x^4}{x^2-1}\,dx = \int \left(x^2+1+\frac{1}{x^2-1}\right)dx.$$

$$= \int x^2 dx + \int 1\,dx + \int \frac{1}{x^2-1}dx = \frac{x^3}{3} + x + \int \frac{1}{(x-1)(x+1)}dx.$$

Also

$$\int \frac{1}{x^2-1}dx = \frac{1}{2}\int \left(\frac{1}{x-1} - \frac{1}{x+1}\right)dx = \frac{1}{2}\left(\ln|x-1| - \ln|x+1|\right)$$

$$= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|.$$

Therefore

$$\boxed{\int \frac{x^4}{x^2-1}\,dx = \frac{x^3}{3} + x + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C}$$

Primitives of functions that can be reduced to rational fractions

$\ast$ Primitives of the Form

$$\int f(\sin x, \cos x)dx.$$

Several methods exists for computing the integral $\int f(\sin x, \cos x)dx$. One of which is completely, although not always the most efficient. We make the change of variable

$$t = \tan\frac{x}{2} \implies dt = \frac{1}{2}(1+\tan^2(\frac{x}{2}))dx = \frac{1}{2}(1+t^2)dx$$

. Then

$$dx = \frac{2dt}{1+t^2}$$

then we have :

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad \tan x = \frac{2t}{1-t^2}, \quad dx = \frac{2}{1+t^2}dt.$$

Therefore the integral becomes :

$$\int f\big(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\big)\frac{2}{1+t^2}dt$$

Example 17

We want to calculate the following integral

$$\int \frac{\sin x}{1-\sin x}dx.$$

By previous change variable, we get

$$\int \frac{\sin x}{1-\sin x}\,dx = \int \frac{\frac{2t}{1+t^2}}{1-\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt = \int \frac{4t}{(t-1)^2(1+t^2)}\,dt.$$

$$\frac{4t}{(t-1)^2(1+t^2)} = \frac{A}{t-1} + \frac{B}{(t-1)^2} + \frac{Ct+D}{1+t^2}.$$

By identification, we find :

$$A = 0, \qquad B = 2, \qquad C = 0, \qquad D = -2. \quad \textit{Check}$$

$$\frac{4t}{(t-1)^2(1+t^2)} = \frac{2}{(t-1)^2} - \frac{2}{1+t^2}.$$

$$\int \frac{\sin x}{1-\sin x}\,dx = \int \left(\frac{2}{(t-1)^2} - \frac{2}{1+t^2}\right) dt = -\frac{2}{t-1} + 2\arctan t + C$$

Back substitution :

$$\int \frac{\sin x}{1-\sin x}\,dx = -\frac{2}{\tan\frac{x}{2}-1} - 2\arctan\left(\tan\frac{x}{2}\right) + C = -\frac{2}{\tan\frac{x}{2}-1} - x + C.$$

Primitives of the Form $\int f(\sin x, \cos x)dx$

Example 18

We search the primitive of the function f defined by

$$f(x) = \frac{1}{3 + \sin x}$$

$$I = \int \frac{1}{3 + \frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\, dt == \frac{3+3t^2+2t}{1+t^2}.$$

$$I = \int \frac{2}{3+3t^2+2t}\, dt.$$

We can write the denominator as the form :

$$3t^2+2t+3 = 3\left(t^2+\frac{2}{3}t+1\right) = 3\left[\left(t+\frac{1}{3}\right)^2+\frac{8}{9}\right].$$

$$I = \int \frac{2}{3\left[\left(t+\frac{1}{3}\right)^2+\frac{8}{9}\right]} dt = \frac{2}{3}\int \frac{dt}{\left(t+\frac{1}{3}\right)^2+\left(\frac{2\sqrt{2}}{3}\right)^2}.$$

Using

$$\int \frac{du}{u^2 + a^2} = \frac{1}{a} \arctan \frac{u}{a},$$

we obtain :

$$I = \frac{2}{3} \cdot \frac{3}{2\sqrt{2}} \arctan\left(\frac{t + \frac{1}{3}}{\frac{2\sqrt{2}}{3}}\right) + C = \frac{1}{\sqrt{2}} \arctan\left(\frac{3t + 1}{2\sqrt{2}}\right) + C.$$

Substitute $t = \tan \frac{x}{2}$:

$$\int \frac{1}{3 + \sin x}\, dx = \frac{1}{\sqrt{2}} \arctan\left(\frac{3\tan \frac{x}{2} + 1}{2\sqrt{2}}\right) + C.$$

Exercise

You can use the change variable $t = \tan x$. Calculate

$$I = \int \frac{dx}{(\sin x + \cos x)^2}\, dx; \quad J = \int \frac{dx}{2\sin^2 3x - 3\cos^2 3x + 1}\, dx;$$

We make $t = e^x \implies dt = tdx$ then

$$\int f(e^x)dx = \int f(t)\frac{dt}{t}.$$

Example 19

Calculate :

$$\int \frac{e^x}{1+e^{3x}}, \qquad \int \frac{shx}{chx+shx}dx.$$

$$\int \frac{e^x}{1+e^{3x}} = \int \frac{t}{1+t^3} = \frac{t}{(t+1)(t^2-t+1)}dt.$$

The partial fractions are :

$$\frac{1}{1+t^3} = \frac{1}{3}\frac{1}{t+1} + \frac{-\frac{1}{3}t+\frac{2}{3}}{t^2-t+1}.$$

Thus

$$\int \frac{dt}{1+t^3} = \frac{1}{3}\int \frac{dt}{t+1} - \frac{1}{3}\int \frac{t\,dt}{t^2-t+1} + \frac{2}{3}\int \frac{dt}{t^2-t+1}.$$

Now,

$$\int \frac{dt}{t+1} = \ln|t+1|,$$

$$\int \frac{t\,dt}{t^2-t+1} = \frac{1}{2}\ln(t^2-t+1) + \frac{1}{\sqrt{3}}\arctan\left(\frac{2t-1}{\sqrt{3}}\right),$$

Primitives of the Form $\int f(e^x)dx$, or $\int f(chx, shx)dx$.

$$\int \frac{dt}{t^2 - t + 1} = \frac{2}{\sqrt{3}} \arctan\left(\frac{2t-1}{\sqrt{3}}\right).$$

Therefore,

$$\int \frac{dt}{1+t^3} = \frac{1}{3}\ln|t+1| - \frac{1}{6}\ln(t^2 - t + 1) + \frac{1}{\sqrt{3}} \arctan\left(\frac{2t-1}{\sqrt{3}}\right) + C.$$

Because we have $t = e^x$, then

$$\boxed{\int \frac{e^x}{1+e^{3x}}\,dx = \frac{1}{3}\ln(1+e^x) - \frac{1}{6}\ln(e^{2x} - e^x + 1) + \frac{1}{\sqrt{3}} \arctan\left(\frac{2e^x-1}{\sqrt{3}}\right) + C}$$

$$\int \frac{shx}{chx + shx}dx. \quad \text{We have} \quad chx = \frac{e^x + e^{-x}}{2},\ shx = \frac{e^x - e^{-x}}{2}$$

Then

$$chx + shx = e^x$$

Hence

$$\frac{shx}{chx + shx} = \frac{e^x - e^{-x}}{2e^x} = \frac{1}{2}(1 - e^{-2x}).$$

$$\boxed{\int \frac{shx}{chx + shx}\,dx = \frac{1}{2}x + \frac{1}{4e^{2x}} = \frac{x}{2} + \frac{1}{4}e^{-2x} + C}$$

- Primitives of the Form

$$I = \int f(x, \sqrt{ax^2 + bx + c}).$$

In this case we distinguish three cases :

1. **1-st case** $\Delta > 0$. So let α, β the roots of $ax^2 + bx + c = 0$. we put $\sqrt{ax^2 + bx + c} = (x - \alpha)t \Rightarrow x = \dfrac{a\beta - \alpha t^2}{a - t^2}$.
   Then
   $$I = \int f\left(\frac{a\beta - \alpha t^2}{a - t^2}, t\left(\frac{a\beta - \alpha t^2}{a - t^2} - \alpha\right)\right)dt \qquad \textit{check}$$
   which is a primitive of rational function in terms of t.

Example 20

Computing the integral $J = \int \dfrac{dx}{\sqrt{x^2 + 3x - 4}}$

$$x^2 + 3x - 4 = (x+4)(x-1) = (x+4)^2 t^2.$$

So :

$$(x-1) = (x+4)t^2 \iff x = \frac{1+4t^2}{1-t^2} \iff dx = \frac{10t}{(1-t^2)^2}\, dt$$

$$J = \int \frac{\frac{10t}{(1-t^2)^2}}{\frac{5t}{1-t^2}}\, dt = \ln\left|\frac{1+t}{1-t}\right| + C$$

check

Primitives of the Form $I = \int f(x, \sqrt{ax^2 + bx + c})$.

1. **2-nd case**

$$\Delta = 0 : \begin{cases} a \geq 0, & \text{obvious as first case} \\ a < 0, & \text{impossible to resolve.} \end{cases}$$

2. **3-rd case**

$$\Delta < 0 : \begin{cases} a < 0, & \text{impossible to resolve.} \\ a > 0, & \text{we make the change variable} \\ \sqrt{ax^2 + bx + c} = \sqrt{a}x + t \Rightarrow x = \dfrac{t^2 - c}{b - 2\sqrt{a}t} \end{cases}$$

Example 21

$$I=\int\frac{\sqrt{x^2+x+1}}{x}dx.$$

We have $\Delta=-3<0,\ a=1>0$, we put $\sqrt{x^2+x+1}=x+t\Rightarrow x=\dfrac{1-t^2}{2t-1}$ et $dx=\dfrac{-t^2+2t-2}{(2t-1)^2}dt$ then the integral I becomes

$$I=-2\int\frac{(t^2-t+1)^2}{(2t-1)^2(t^2-1)}dt$$

Solution of example 21

We have

$$(t^2 - t + 1)^2 = t^4 - 2t^3 + 3t^2 - 2t + 1 \quad \textit{check}$$

and

$$(2t-1)^2(t^2-1) = (4t^2 - 4t + 1)(t^2 - 1) = 4t^4 - 4t^3 - 3t^2 + 4t - 1 \quad \textit{check}$$

By euclidean division, we obtain :

$$\frac{t^4 - 2t^3 + 3t^2 - 2t + 1}{4t^4 - 4t^3 - 3t^2 + 4t - 1} = \frac{1}{4} + \frac{-t^3 + \frac{15}{4}t^2 - 3t + \frac{5}{4}}{(2t-1)^2(t^2-1)} \quad \textit{check}$$

Now decompose the remainder like this :

$$\frac{-t^3 + \frac{15}{4}t^2 - 3t + \frac{5}{4}}{(2t-1)^2(t^2-1)} = \frac{A}{t-1} + \frac{B}{t+1} + \frac{C}{2t-1} + \frac{D}{(2t-1)^2}$$

Solution of example 21

By identification we find

$$A = \frac{1}{2}, \qquad B = -\frac{1}{2}, \qquad C = \frac{1}{2}, \qquad D = -\frac{1}{2}. \quad \textit{check}$$

Hence

$$\frac{-t^3 + \frac{15}{4}t^2 - 3t + \frac{5}{4}}{(2t-1)^2(t^2-1)} = \frac{1}{2(t-1)} - \frac{1}{2(t+1)} + \frac{1}{2(2t-1)} - \frac{1}{2(2t-1)^2}$$

Therefore

$$I = -\frac{1}{2}t - 2\int \left(\frac{1}{2(t-1)} - \frac{1}{2(t+1)} + \frac{1}{2(2t-1)} - \frac{2}{2(2t-1)^2} \right) dt$$

Integrating term by term :

$$I = -\frac{1}{2}t - \ln|t-1| + 1\ln|t+1| - \ln|2t-1| - \frac{1}{2t-1} + C.$$

Solution of example 21

Finally

$$I=-\frac{1}{2}(\sqrt{x^2+x+1}-x)-\ln|\sqrt{x^2+x+1}-x-1|+\ln|\sqrt{x^2+x+1}-x+1|$$

$$-\ln|2\sqrt{x^2+x+1}-2x-1|-\frac{1}{2\sqrt{x^2+x+1}-2x-1}+C$$

Exercise

Find the primitive function J and K such that

$$J=\int\sqrt{x^2+1}\,dx;\quad K=\int\sqrt{x^2-4}\,dx.$$

Solution

We make :

$$\sqrt{x^2+1} = x + t \implies x = \frac{1-t^2}{2t} \quad \text{and } dx = -\frac{t^2+1}{2t^2}dt$$

$$J = -\int \frac{t^4+2t^2+1}{4t^3}\,dt = -\frac{1}{8}t^2 - \frac{1}{2}\ln|t| - \frac{1}{8t^2} + C$$

$$J = -\frac{1}{8}(\sqrt{x^2+1}-x)^2 - \frac{1}{2}\ln|\sqrt{x^2+1}-x| - \frac{1}{8(\sqrt{x^2+1}-x)^2} + C$$

Solution

Or we can use the change variable

$$x = Sh(t) \implies dx = Ch(t)dt$$

, so

$$J = \int Ch^2(t)\,dt = \frac{1}{2}\int (Ch(2t)+1)dt = \frac{1}{4}Sh(2t) + \frac{t}{2} + C$$

Therefore

$$J = \frac{1}{4}Sh(2Argshx) + \frac{1}{2}Argshx + C$$

Solution

For the second integral, first method we use the change variable

$$\frac{x}{2} = Ch(t) \implies dx = 2Sh(t)dt$$

, so

$$K = \int Ch^2(t)\,dt = \frac{1}{2}\int (Ch(2t)+1)dt = \frac{1}{4}Sh(2t) + \frac{t}{2} + C$$

Then

$$\sqrt{x^2-4} = \sqrt{4Ch^2(t)-4} = 2\sqrt{Ch^2(t)-1} = 2Sh(t).$$

Thus

$$K = \int (2Sh(t))(2Sh(t))\,dt = 4\int Sh^2(t)\,dt = 2\int (Ch(2t)-1)\,dt..$$

Hence

$$K = 2\left(\frac{1}{2}Sh(2t) - t\right) + C = Sh(2t) - 2t + C.$$

Solution

Now return to x. Since

$$Sh(2t) = 2Sh(t)Ch(t),$$

and

$$Cht = \frac{x}{2}, \qquad Sht = \sqrt{Ch^2t - 1} = \frac{\sqrt{x^2-4}}{2},$$

we get

$$Sh(2t) = \frac{x\sqrt{x^2-4}}{2}.$$

Also

$$t = \text{Argch}\left(\frac{x}{2}\right) = \ln\left(\frac{x+\sqrt{x^2-4}}{2}\right).$$

Therefore

$$K = \frac{x}{2}\sqrt{x^2-4} - 2\ln\left(\frac{x+\sqrt{x^2-4}}{2}\right) + C.$$

Solution

Or, we set up $\sqrt{(x-2)(x+2)} = (x+2)t$ then

$$x = \frac{2(1+t^2)}{1-t^2} \implies dx = \frac{8t}{(1-t^2)^2}dt$$

Hence

$$K = 32\int \frac{t^2}{(1-t^2)^3}\,dt$$

Write

$$\frac{t^2}{(1-t^2)^3} = \frac{1}{(1-t^2)^3} - \frac{1}{(1-t^2)^2}.$$

Hence

$$K = \int \frac{1}{(1-t^2)^3}dt - \int \frac{1}{(1-t^2)^2}dt.$$

First compute

$$\int \frac{1}{(1-t^2)^2}dt = \frac{t}{2(1-t^2)} + \frac{1}{4}\ln\left|\frac{1+t}{1-t}\right| + C. \quad \textit{check}$$

Solution

Next compute

$$\int \frac{1}{(1-t^2)^3}dt = \frac{t}{4(1-t^2)^2} + \frac{3t}{8(1-t^2)} + \frac{3}{16}\ln\left|\frac{1+t}{1-t}\right| + C. \quad \textit{check}$$

Therefore

$$K = \left(\frac{t}{4(1-t^2)^2} + \frac{3t}{8(1-t^2)} + \frac{3}{16}\ln\left|\frac{1+t}{1-t}\right|\right) - \left(\frac{t}{2(1-t^2)} + \frac{1}{4}\ln\left|\frac{1+t}{1-t}\right|\right) + C.$$

After simplification we obtain

$$K = \frac{t}{4(1-t^2)^2} - \frac{t}{8(1-t^2)} - \frac{1}{16}\ln\left|\frac{1+t}{1-t}\right| + C.$$

Using the change variable

$$t = \sqrt[n]{\frac{ax+b}{cx+d}} \Longrightarrow t^n = \frac{ax+b}{cx+d}$$

We treat this case with the following example

Example 22

Calculate

$$I = \int \sqrt{\frac{1+x}{1-x}}\,\frac{dx}{x}$$

Put

$$t^2 = \frac{1+x}{1-x} \implies x = \frac{t^2-1}{t^2+1},$$

Thus

$$dx = \frac{4t^3}{(t^2+1)^2}$$

Then I become a rational fraction

$$I = \int \frac{4t^4}{t^4-1}\, dt$$

The student must complete the solution of fraction integral as previous methods.

Primitives of the Form $\int f(x^{(\frac{m_1}{n_1},...,\frac{m_k}{n_k})})dx$

Let k be the least common factor of the denominators $n_1, ..., n_k$, and we use the change variable

$$x = t^k \implies dx = kt^{k-1}dt$$

Example 23

Compute

$$\int \frac{x^{\frac{1}{2}}}{x^{\frac{1}{4}} + 1}\, dx$$

We have $k = 4$ then $x = t^4 \implies dx = 4t^3 dt$ and the integral become integral of rational fraction as follow

$$\int \frac{t^2}{t^3 + 1}\, dt$$

The student must complete the solution of fraction integral as previous methods.

End Chapter 2