The Finite Expansions

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The Finite Expansion

Introduction

Under a certain conditions, a function can be approximated near a given point by a polynomial. If the differentiability conditions are satisfied, we can write a Taylor expansion near that point. However, even if a function does not satisfy all differentiability conditions, it is sometimes possible to write an approximate polynomial expansion in a neighborhood of a point.
This idea leads us to the concept of **finite expansion**.

Londau Notation

Definition

let $x_0 \in \mathbb{R}$ and f, g tow continuous functions defines in the neighborhood to x_0. We say that f is negligible compared to g around to x_0 if $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} = 0$.
We note $f << g$ (notation de Hardy) or
$f = o(g)$ (notation de Londau)

Remark

- $f = o(1) \Rightarrow \lim\limits_{x \to x_0} f = 0$.
- $f \mathfrak{R} g \Leftrightarrow f = o(g)$ is not equivalent relation (because it is transitive, but not reflexive and not symmetric).
- If $\alpha < \beta \Rightarrow x^\alpha = o(x^\beta)$.

Polynomial Approximation

The finite expansions essentially to find a polynomial approximation to a more complicated function in the neighborhood of a some point. They have numerous applications in other sciences, but also in mathematics itself, particularly in numerical analysis.

Definition

We said that $f : I \to \mathbb{R}$, is represented by the polynomial approximation of degree n, for x near to $x_0 \in I$ if and only if there exist a polynomial $P \in \mathbb{R}_n[X]$, such that

$$\forall x \in I : f(x) = P(x - x_0) + o(x - x_0)^n.$$

We call $P(x - x_0)$ is **the mean part** of the finite expansions and $o(x - x_0)^n$ is **the remainder part (or, rest)** of degree n.

Taylor Polynomial

The approximation of the function f by the Taylor polynomial of degree n, denoted by $P(x-x_0)$ for x near to x_0 using more derivatives $f'(x), f''(x), ..., f^n(x)$ is given by

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + ...$$

$$+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + O(x-x_0)^n$$

Particular case. if $x = 0$.

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + ... + \frac{f^{(n)}(0)}{n!}x^n + O(x^n)$$

This formula is called the formula of **Mac-Laurin**

Example

We will write the Mac-Laurin formula of the function $\cos : x \mapsto \cos x$ near to 0. of degree n

$$\cos' x = -\sin x,$$

$$\cos'' x = -\cos x,$$

$$\cos^{(3)} x = \sin x,$$

$$\cos^{(4)} x = \cos x,$$

$$\cos^{(5)} x = -\sin x,$$

$$\cos^{(5)} x = -\cos x,$$

It's easy to find that,

$$\cos^{(n)}(x) = \cos(x + \frac{n\pi}{2})$$

Then

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + ... + (-1)^n \frac{x^{2n}}{(2n)!} + O(x^n)$$

Example

The polynomial approximation of the function cos of degree 4 is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + O(x^4),$$

The polynomial approximation of degree 5 is given by :

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + O(x^5),$$

The first statement informs us that there are terms of order x^4 in the expansion. The second statement is stronger as it informs us that there are no terms of order x^5.

Remark

The remainder term $O(x^n)$ can also be written using a negligible function ϵ, such that $x^n\epsilon(x) = O(x^n)$ and $\lim_{x\to 0}\epsilon(x) = 0$.

Finite Expansion at zero

Definition

Let f be a real valued function. We said that the function f is represented by a finite expansion at zero if there exist real numbers $a_0, a_1, ...a_n$ and a real valued function ϵ such that

$$f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + x^n \epsilon(x), \qquad \lim_{x \to 0} \epsilon(x) = 0.$$

Then the function f is represented by the polynomial approximation of degree n, denoted by $P_n(x)$ for x near zero, which is called the main part of finite expansions at zero, such that :
$P_n(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$

Example

1. Let the function f defined by

$$f(x) = x + x^2 + x^6 \cos\left(\frac{1}{x}\right).$$

This function admits a finite expansion of order 5 near zero, since it can be written as :

$$f(x) = x + x^2 + x^5\left(x\cos(\frac{1}{x})\right) = x + x^2 + x^5\epsilon(x) \quad \lim_{x\to 0}\epsilon(x) = 0.$$

2. Using the euclidean division by increasing power order, one has the finite expansion at zero of

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + ... + x^n + \frac{x^{n+1}}{1-x}$$
$$= 1 + x + x^2 + x^3 + ... + x^n + x^n(\frac{x}{1-x})$$

in this case $\epsilon(x) = \dfrac{x}{1-x}$. We generally do not try to determine the function $\epsilon(x)$.

Properties

Proposition

If a real function f admits a finite expansion of order n in a neighborhood of zero, then this expansion is unique.

Proof.

Suppose that the function f admits two finite expansion of order n in a neighborhood of zero

$$f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n + x^n\epsilon_1(x) \quad (1)$$

$$f(x) = ab_0 + b_1x + b_2x^2 + ... + b_nx^n + x^n\epsilon_2(x) \quad (2)$$

We want to show that : $a_k = b_k, \quad \forall k = 0, 1, ..., n.$

We proceed by contradiction. Assume the apposite, and let r be the smallest integer such that : $a_r \neq b_r$.

Subtracting equations (1) and (2), we obtain :

$$0 = (a_r - b_r)x^r + (a_{r+1} - b_{r+1})x^{r+1} + ... + (a_n - b_n)x^n x^n(\epsilon_1(x) - \epsilon_2(x)).$$

Properties

Dividing both sides by x^r, we get :

$$0 = (a_r - b_r) + (a_{r+1} - b_{r+1})x + \ldots + (a_n - b_n)x^{n-r} + x^{n-r}(\epsilon_1(x) - \epsilon_2(x)).$$

Letting $x \to 0$ we obtain $a_r - b_r = 0 \implies a_r = b_r$

Lemma

If a real function f can be expanded at zero, then $\lim_{x \to 0} f(x) = a_0$. If f continuous at 0, we obtain :

$$\lim_{x \to 0} f(x) = a_0 = f(0).$$

Finally, if $n \geq 1$ the function f is continuous at 0 and differentiable at 0, and we have : $f'(0) = a$

Properties

Proof

From the finite expansion of f, it immediately follows that :

$$\lim_{x\to 0} f(x) = a_0.$$

Therefore, if f is continuous at 0, we can write :

$$\lim_{x\to 0} f(x) = a_0 = f(0).$$

Now, from the differentiability at 0, we write :

$$\frac{f(x)-f(0)}{x} = a_1 + a_2 x + \ldots + a_n x^{n-1} + x^{n-1}\epsilon(x)$$

Hence

$$f'(0) = \frac{f(x)-f(0)}{x} = a_1.$$

Remark

From this lemma, we deduce that if a function f is differentiable at 0 and admits a finite expansion in a neighborhood of 0, then the coefficient a_1 is zero if and only if the graph of f has a tangent at 0 parallel to the x-axis.

Properties

Proposition

If the function f is odd and admits a finite expansion of order n near zero :

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + x^n \epsilon(x).$$

Then :

$a_0 = a_2 = \ldots = a_{2r} = 0, \quad \forall r \in \mathbb{N}$ such that $2r \leq n$.

If f is even, then :

$a_1 = a_3 = \ldots = a_{2r+1} = 0, \quad \forall r \in \mathbb{N}$ such that $2r+1 \leq n$.

proof

In the first cases, we have :

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + x^n \epsilon(x)$$

$$f(-x) = a_0 - a_1 x + a_2 x^2 + \ldots + (-1)^n a_n x^n + x^n \epsilon_1(x)$$

$$-f(-x) = -a_0 + a_1 x - a_2 x^2 + \ldots + (-1)^{n+1} a_n x^n + x^n \epsilon_2(x)$$

Properties

Such that $\lim_{x\to 0}\epsilon(x) = \lim_{x\to 0}\epsilon_1(x) = \lim_{x\to 0}\epsilon_2(x) = 0$
Since $f(x) = -f(-x)$ and the finite expansion is unique
$a_0 = a_2 = \ldots = a_{2r} = 0, \quad \forall r \in \mathbb{N}$ such that $2r \leq n$.
The second case is treated in the same manner.

Proposition

If a real function f admits a finite expansion of order n in a neighborhood of zero, then it admits a finite expansion of any order $m \leq n$ in a neighborhood of zero.

Properties

proof

Suppose

$$f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n + x^n\epsilon(x)$$

We can rewrite this as :

$$f(x) = a_0 + a_1x + a_2x^2 + ... + a_mx^m + a_{m+1}x^{m+1} + a_nx^n + x^n\epsilon(x)$$

Factoring x^m we obtain :

$$f(x) = a_0+a_1x+a_2x^2+...+a_mx^m+x^m(a_{m+1}x+..+a_nx^{n-m}+x^{n-m}\epsilon(x)).$$

Let us define :

$$\epsilon_m(x) = a_{m+1}x + ... + a_nx^{n-m} + x^{n-m}\epsilon(x).$$

Since $\lim_{x\to 0}\epsilon_m(x) = 0$,

$$f(x) = a_0 + a_1x + a_2x^2 + ... + a_mx^m + x^m\epsilon_m(x)).$$

Thus, f admits a finite expansion of order m near zero.

Finite expansions of elementary functions

The function defined by : $f(x) = a^x,\ a > 0.$ We have $a^x = e^{x \ln a}$ and the n-th derivative of f is $(a^x)^{(n)} = (\ln a)^n e^{x \ln a}$. So the finite expansions of this function is given by

$$1.)\quad a^x = 1 + (\ln a)x + \frac{(\ln a)^2}{2!}x^2 + \dots + \frac{(\ln a)^n}{n!}x^n + x^n\epsilon(x)$$

For $a = e$

$$2.)\quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + x^n\epsilon(x)$$

$f(x) = Ch(x)$, the higher derivatives functions of f is

$$(Ch(x))^n = \begin{cases} Ch(x); & n \text{ even} \\ Sh(x); & n \text{ odd} \end{cases}$$

$$Ch(x) = Ch(0) + \frac{Ch'(0)}{1!}x + \frac{Ch''(0)}{2!}x^2 + \dots + \frac{Ch^{(n)}(0)}{n!}x^n + x^n\epsilon(x)$$

$$3.)\quad Ch(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{x^{2k}}{(2k)!} + x^{2k}\epsilon(x);\quad (2k \le n)$$

Finite expansions of elementary functions

Is the same, we have for the function $f(x) = Sh(x)$, the higher derivatives functions of f is

$$(Sh(x))^n = \begin{cases} Sh(x); & n \text{ } even \\ Ch(x); & n \text{ } odd \end{cases}$$

So

$$Sh(x) = Sh(0) + \frac{Sh'(0)}{1!}x + \frac{Sh''(0)}{2!}x^2 + ... + \frac{Sh^{(n)}(0)}{n!}x^n + x^n\epsilon(x)$$

4.) $$Sh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ... + \frac{x^{2n+1}}{(2n+1)!} + x^{2n+1}\epsilon(x); \quad (2k+1 \leq n)$$

$$\cos(x) = \cos(0) + \frac{\cos'(0)}{1!}x + \frac{\cos''(0)}{2!}x^2 + ... + \frac{\cos^{(n)}(0)}{n!}x^n + x^n\epsilon(x)$$

5.) $$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ... + (-1)^n\frac{x^{2k}}{(2k)!} + x^{2k}\epsilon(x) \quad (2k \leq n)$$

Finite expansions of elementary functions

6.) $$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + (-1)^n \frac{x^{2k+1}}{(2k+1)!} + x^{2k+1}\epsilon(x) \quad (2k+1 \leq n)$$

7.) $$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots + (-1)^n x^n + x^n \epsilon(x)$$

8.) $$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n-1}\frac{x^n}{n} + x^n \epsilon(x)$$

For $\alpha \in \mathbb{R}$ et $x \neq -1$

9.) $$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \ldots + \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}x^n + x^n \epsilon($$

For $\alpha = \frac{1}{2}$

10.) $$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ldots + (-1)^{n+1}\frac{1.3.5\ldots(2n-3)}{2^n n!}x^n + x^n$$

Finite expansions of elementary functions

By replacing x with $-x$ we get

$$11.) \quad \sqrt{1-x} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - ... - \frac{1.3.5...(2n-3)}{2^n n!}x^n + x^n \epsilon(x)$$

For $\alpha = -1$ we fall back on the finite expansion of $\dfrac{1}{1+x}$

$$12.) \quad \frac{1}{1-x} = 1 + x + x^2 + x^3 + ... + x^n + x^n \epsilon(x)$$

Remark

We know that if a function f admits continuous derivatives functions $f', f'', ..., f^{(n)}$ and a bounded derivative $f^{(n+1)}$ the Mac-Laurin polynomial given by

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + ... + \frac{f^{(n)}(0)}{n!}x^n + x^n\epsilon(x) \quad (1)$$

Moreover, these hypothesis allow us to write Mac-Laurin formula of order $n-1$ for the derivative f'.

$$f'(x) = f'(0) + \frac{f''(0)}{1!}x + \frac{f^{(3)}(0)}{2!}x^2 + ... + \frac{f^{(n)}(0)}{(n-1)!}x^{n-1} + x^{n-1}\epsilon(x) \quad (2)$$

If we compare these two expansions, we observe that the main part in (2) is the derivative of the main in (1).

Thus, if the Mac-laurin polynomial applied to function f gives :

$$f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + x^n \epsilon(x),$$

then the the Mac-laurin polynomial of its derivative is :

$$f'(x) = a_1 + 2.a_2 x^2 + ... + n.a_n x^{n-1} + x^{n-1} \epsilon(x),$$

and vice versa (conversely), if Taylor's expansion of the derivative f' is known :

$$f'(x) = b_1 + b_2 x + ... + a_n x^{n-1} + x^{n-1} \epsilon(x),$$

then we can deduce the expansion of the function

$$f(x) = b_1 x + \frac{b_2}{2} x^2 + ... + \frac{a_n}{n} x^n + x^n \epsilon(x),$$

This remark can be applied in the following cases :

title

a We have

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots + x^n + x^n\epsilon(x)$$

Therefore

$$g(x) = f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots + nx^{n-1} + x^{n-1}\epsilon(x)$$

b We have

$$f(x) = \frac{1}{1+x} = 1 - x + x^2 + \ldots + (-1)^n x^n + x^n\epsilon(x)$$

Therefore

$$g(x) = f'(x) = \frac{-1}{(1+x)^2} = -1 + 2x - 3x^2 + \ldots + (-1)^n nx^{n-1} + x^{n-1}\epsilon(x$$

c We have

$$f(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 + ... + (-1)^n x^{2n} + x^n \epsilon(x)$$

Therefore

$$g(x) = \arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} + ... + (-1)^n \frac{x^{2n+1}}{2n+1} + x^{2n+1}\epsilon(x)$$

(Note that : $\arctan(0) = 0$.)

d We have

$$f(x) = \frac{1}{1+x} = 1 - x + x^2 + ... + (-1)^n x^n + x^n \epsilon(x)$$

Therefore

$$g(x) = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + ... + (-1)^n \frac{x^{n+1}}{n+1} + x^{n+1}\epsilon(x)$$

Algebraic combinations of finite expansion

If f and g can both be expanded at zero and λ is any constant, then each of the following functions is also can be expanded at zero : The sum $f + g$, the difference $f - g$, the constant multiple $\lambda \times f$, the product $f \times g$, the quotient $f \div g$, if $g(0) \neq 0$: and composite $f \circ g$
Consider the finite expansion at zero of f and g

$$f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + x^n \epsilon_1(x)$$

and

$$g(x) = b_0 + b_1 x + b_2 x^2 + ... + b_n x^n + x^n \epsilon_2(x)$$

such that $\lim\limits_{x \to 0} \epsilon_1(x) = 0, \quad \lim\limits_{x \to 0} \epsilon_2(x) = 0$ then the finite expansion of

The sum rule $f+g$

the sum $f+g$ is

$$(f+g)(x)=f(x)+g(x)=(a_0+a_1x+a_2x^2+...+a_nx^n)$$

$$+(b_0+b_1x+b_2x^2+...+b_nx^n)+x^n(\epsilon_1(x)+\epsilon_2(x))$$

Therefore

$$(f+g)(x)=(a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2+...+(a_n+b_n)x^n$$

$$+x^n(\epsilon_1(x)+\epsilon_2(x))$$

and $\lim_{x\to 0}(\epsilon_1(x)+\epsilon_2(x))=0$

For example : If $f(x)=e^x$ and $g(x)=e^{-x}$ then

$$e^x+e^{-x}=2+2\frac{x^2}{2!}+2\frac{x^4}{4!}+...+2\frac{x^{2k}}{(2k)!}+x^{2k}\epsilon(x) \quad 2k\le n$$

Hence

$$\frac{e^x+e^{-x}}{2}=1+\frac{x^2}{2!}+\frac{x^4}{4!}+...+\frac{x^{2k}}{(2k)!}+x^{2k}\epsilon(x)=Ch(x) \quad 2k\le n$$

The product rule $f.g$

The finite expansion at zero of the $f.g$ is obtained by the product

$$(a_0 + a_1x + a_2x^2 + ... + a_nx^n)(b_0 + b_1x + b_2x^2 + ... + b_nx^n)$$

and keeping only the monomials of degree less than or equal n in the product

$$(f.g)(x) = f(x).g(x) = (A(x) + x^n\epsilon_1(x)) + (B(x) + x^n\epsilon_2(x))$$

such that

$$A(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$$

and

$$B(x) = b_0 + b_1x + b_2x^2 + ... + b_nx^n$$

Then

$$(f.g)(x) = A(x).B(x) + x^n\epsilon_1(x).B(x) + x^n\epsilon_2(x).B(x) + x^{2n}\epsilon_1(x).\epsilon_2(x)$$

The product rule $f.g$

If we denote by $C(x)$ the sum terms whose degree is less than or equal to n, we write :

$$A(x).B(x) = C(x) + x^{n+1}D(x),$$

hence

$$A(x).B(x) = C(x)x^n(\epsilon_1(x).B(x)+\epsilon_2(x).A(x)+x^n\epsilon_1(x).\epsilon_2(x)+xD(x)).$$

Then

$$f(x).g(x) = C(x) + x^n\epsilon(x),$$

such that

$$\lim_{x\to 0}\epsilon(x) = \lim_{x\to 0}(\epsilon_1(x).B(x)+\epsilon_2(x).A(x)+x^n\epsilon_1(x).\epsilon_2(x)+xD(x)) = 0$$

The product rule $f.g$

Example

Let the function $f(x) = e^x.\sin(x)$. We went to find the finite expansion of the function f of degree 6 near to zero.

$$f(x) = e^x.\sin(x) = (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + x^6\epsilon_1(x)).$$

$$(x - \frac{x^3}{3!} + \frac{x^5}{5!} + x^n\epsilon_2(x))$$

$$f(x) = x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \frac{x^6}{90} + x^6\epsilon(x).$$

This result was obtained in practice by the usual multiplication of the two regular parts associated with two functions. and keeping only the terms whose degree does not exceed 6.

Question : Answer the same question concerning the function f defined by : $f(x) = \dfrac{\ln(1+x)}{1+x}$.

The division f/g

The finite expansion at zero of the quotient f/g such that ($b_0 \neq 0$ this ensures that the function g is not nullified) is obtained by the euclidean division of

$$(a_0 + a_1x + a_2x^2 + ... + a_nx^n)$$

by

$$(b_0 + b_1x + b_2x^2 + ... + b_nx^n)$$

by increasing power order.

$$A(x) = B(x).Q(x) + x^{n+1}R(x), \text{ with } deg(Q) \leq n.$$

Such that Q is mean part of the finite expansion of order n of the function $\dfrac{f}{g}$ and $xR(x) = \epsilon(x)$

Therefore

$$f(x) - x^n\epsilon_1(x) = (g(x) - x^n\epsilon_2(x))Q(x) + x^n\epsilon(x)$$

$$\frac{f(x)}{g(x)} = Q(x) + x^n\left(\frac{\epsilon(x) + \epsilon_1(x) - \epsilon_2(x)}{g(x)}\right)$$

The division f/g

Example

The finite expansions of degree 5 of the function $th : x \mapsto th(x)$

$$th(x) = \frac{sh(x)}{ch(x)} = \frac{x + \frac{x^3}{3!} + \frac{x^5}{5!} + x^5\epsilon(x)}{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + x^5\epsilon(x)}$$

$$= x - \frac{x^3}{3} + \frac{2x^5}{15} + \frac{x^5.(\frac{x^2}{72} - \frac{x^4}{15} - \frac{x^6}{180})}{1 + \frac{x^2}{2!} + \frac{x^4}{4!}}$$

Thus

$$th(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} + x^5\epsilon(x)$$

The division f/g

We answer to the same question from the function $f : x \mapsto \tan x$. We have

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} + x^5\epsilon_1(x)}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} + x^5\epsilon_2(x)}.$$

And by euclidean division of the two main parts we easy obtain

$$x + \frac{x^3}{3} + \frac{2x^5}{15} + x^5\epsilon(x).$$

So did it

Composite of finite expansion

Let u be a real function constrained by the condition

$$\lim_{x \to 0} u(x) = 0,$$

and has a finite expansion of order n near zero as follow

$$u(x) = B(x) + x^n \epsilon_1(x) = b_1 x + b_2 x^2 + ... + b_n x^n + x^n \epsilon_1(x),$$

And let $f : u \mapsto f(u)$ be function defined a neighborhood to zero and has a finite expansion near to 0 :

$$f(u) = a_0 + a_1 u + a_2 u^2 + ... + a_n u^n + u^n \epsilon_2(u),$$

such that $\lim_{x \to 0} \epsilon_1(x) = \epsilon_2(u) = 0$.
Then the function $h : x \mapsto f(u(x))$ has a finite expansion near zero.
And indeed, we can write :

Composite of finite expansion

$$h(x) = f(u(x)) = a_0 + a_1(B(x) + x^n\epsilon_1(x)) + a_2(B(x) + x^n\epsilon_1(x))^2$$

$$+ ... + a_n(B(x) + x^n\epsilon_1(x))^n + (B(x) + x^n\epsilon_1(x))^n\epsilon_2(u(x)),$$

For example, the finite expansion of degree 2 near 0 of the function $x \mapsto e^{\sin x}$.
we have

$$\sin x = x + x^2\epsilon(x) \quad \text{and } e^x = 1 + x + \frac{x^2}{2} + \epsilon_2(x)$$

then $e^{\sin x} = 1 + x + \dfrac{x^2}{2} + \epsilon(x)$ with $\lim_{x\to 0} \epsilon(x) = 0$.

Example

Example

We expand the function defined by $\ln(\cos x)$ near zero up to the order 4. We have

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + x^4\epsilon_1(x)$$

Assume that

$$u(x) = \cos x - 1 = -\frac{x^2}{2} + \frac{x^4}{24} + x^4\epsilon_1(x)$$

Also

$$\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + u^4\epsilon_2(u)$$

Example

Example

Therefore

$$\ln(\cos x) = (-\frac{x^2}{2} + \frac{x^4}{24}) - \frac{x^4}{8} + x^4\epsilon(x)]$$

$$\ln(\cos x) = -\frac{x^2}{2} - \frac{x^4}{12} + x^4\epsilon(x).$$

Of course, we keep only the terms of order less or equal 4..

• We answer the same question of the function $g(x) = e^{\cos x}$ up to the order 3.

$$e^{u(x)} = e^{\cos x - 1} = 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + u^3\epsilon_2(u)$$

Then

$$e^{\cos x} = e - \frac{ex^2}{2} + x^3\epsilon(x).$$

Check.

The finite expansion at a point

Definition

We said that the function $f : x \mapsto f(x)$ can be represented by a finite expansion near to x_0 if the function $t \mapsto f(t + x_0)$. can be represented by finite expansion to zero of order n. and then we will have

$$f(t) = a_0 + a_1 t + a_2 t^2 + ... + a_n t^n + t^n \epsilon(t),$$

such that $\lim_{t \to 0} \epsilon(t) = 0$. Often we therefore reduce the problem to 0 by changing the variables $t = x - x_0$.
Thus

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + ... + a_n(x - x_0)^n + (x - x_0)^n \epsilon(x - x_0),$$

such that $\lim_{x \to x_0} \epsilon(x - x_0) = 0$.

Example

Example

The finite expansion of $f(x) = e^x$ near to 1 of order n.
We make the change variable $t = x - 1$. If x is near to 1 then t is near to 0.

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots + \frac{t^n}{n!} + t^n \epsilon(t)$$

$$e^x = e(1+(x-1)+\frac{(x-1)^2}{2!}+\frac{(x-1)^3}{3!}+\ldots+\frac{(x-1)^n}{n!}+x^n\epsilon(x-1)),$$

$$\lim_{x\to 1} \epsilon(x-1) = 0.$$

Example

Example

We expand the function $x \mapsto chx$ a neighborhood to 2 of degree 3. We have

$$ch(x) = ch(t+2) = ch(t)ch(2) + sh(t)sh(2)$$

$$ch(2)(1 + \frac{t^2}{2} + t^3\epsilon_1(t)) + sh(t)(t + \frac{t^3}{6} + t^3\epsilon_2(t))$$

Therefore

$$ch(2)(1 + \frac{(x-2)^2}{2} + (x-2)^3\epsilon_1(x-2)) +$$

$$sh(2)((x-2) + \frac{(x-2)^3}{6} + (x-2)^3\epsilon_2(x-2))$$

$$\frac{sh2}{6}x^3 + \frac{e^{-2}}{2}x^2 + (sh2 - e^{-2})x + (3ch2 - \frac{2}{3}sh2) + (x-2)^3\epsilon(x-2).$$

The finite expansion at Infinity

Definition

Let f be a function defined a neighborhood $+\infty$ ($-\infty$ successively). We said that the function $f : x \mapsto f(x)$ has a finite expansion at infinity if the function

$$F : t \mapsto f(\frac{1}{x})$$

has a finite expansion at zero.

So, we write

$$F(t) = a_0 + a_1 t + a_2 t^2 + ... + a_n t^n + t^n \epsilon(t),$$

then

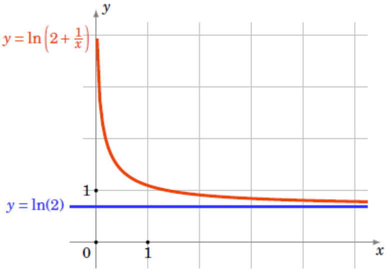
$$f(x) = a_0 + a_1 \frac{1}{x} + a_2 \frac{1}{x^2} + ... + a_n \frac{1}{x^n} + \frac{1}{x^n} \epsilon(\frac{1}{x}),$$

Example

Let the function f defined by : $f : x \mapsto \ln(2 + \dfrac{1}{x})$

$$f(x) = \ln 2 + \ln(1 + \frac{1}{2x}) = \ln 2 + \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{24x^3} + \ldots + \frac{1}{n2^n x^n} + \frac{1}{x^n}\epsilon(\frac{1}{x})$$

This allows us to have a precise idea of the behavior of f in the neighborhood of $+\infty$. When $x \to +\infty$ then $(x) \to \ln 2$. And the second term is $+\dfrac{1}{2}x$ therefore is positive, this means that the function $f(x)$ tends to $\ln 2$ while remaining above to $\ln 2$.



The finite expansion at infinity

Remark

1. The finite expansions at $+\infty$ is also called an asymptotic expansions.
2. We said that the function $x \mapsto f(x)$ can be expended at $+\infty$ of degree n is equivalent to $x \mapsto f(\frac{1}{x})$ can be expended at 0^+ by finite expansions of degree n.
3. We can similarly define what a finite expansion at $-\infty$

Example

Find the finite expansion a neighborhood to infinity of order 4 the real function defined by

$$f(x) = \frac{x^2+1}{x^2-1}$$

By change variable we can write

$$h(t) = \frac{1+t^2}{1-t^2} = (1+t^2)(\frac{1}{1-t^2})$$

$$(1+t^2)(1+t^2+t^4+t^4\epsilon(t)) = 1+2t^2+2t^4+t^4\epsilon(t)$$

Then

$$\frac{x^2+1}{x^2-1} = 1+\frac{2}{x^2}+\frac{2}{x^4}+\frac{1}{x^4}\epsilon(\frac{1}{x}).$$

Generalized finite expansion near zero

Definition

Let f be a function defined in a neighborhood of zero (except possibly at zero). We assume that f does not admit a finite expansion at zero. but there exists a natural number k such that the function $h : x \mapsto x^k f(x)$ has a finite expansion of order n near zero. Thus, for $x \neq 0$:

$$x^k f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + x^n \epsilon(x),$$

therefore

$$f(x) = x^{-k}(a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + x^n \epsilon(x)),$$

We called the last expression, the generalized finite expansion of order n of the function f.

Generalized finite expansion near zero

For example, we take $n = 3$ and $f(x) = \dfrac{e^x}{x}$. We have $\lim_{x\to 0} xf(x) = \lim_{x\to 0} e^x = 1$. Therefore

$$xf(x) = e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + x^4\epsilon(x).$$

Then

$$f(x) = \frac{1}{x} + 1 + \frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + x^3\epsilon(x). \qquad \textit{CHEK}$$

Generalized finite expansion near zero

Answer on the same question for the function $f(x) = \cot x = \dfrac{\cos x}{\sin x}$.

We have : $\lim_{x\to 0} xf(x) = \lim_{x\to 0} \dfrac{x}{\sin x} \cos x = 1$

Therefore

$$xf(x) = \frac{1 - \frac{x^2}{2} + \frac{x^4}{24} + x^4\epsilon_1(x)}{1 - \frac{x^2}{6} + \frac{x^4}{120} + x^4\epsilon_2(x)} = 1 - \frac{x^2}{3} + \frac{x^4}{45} + x^4\epsilon_3(x).$$

Then

$$\cot = \frac{1}{x} - \frac{x}{3} + \frac{x^3}{45} + x^4\epsilon_3(x). \qquad \textit{CHEK}$$

Generalized finite expansion near zero

Exercise : Homework

Bring the generalized finite expansion in both cases :

$$f(x) = \frac{1}{(\sin x)^3}, \quad n = 2,\ x_0 = 0,$$

$$g(x) = \frac{1}{\ln(1+\sin x)}, \quad n = 3,\ x_0 = 0.$$

Applications

Using the finite expansions to evaluate limits

The finite expansions provide a good way to understand the behavior of a function near a specified point and so are useful for solving some indeterminate forms. When taking a limit as $x \to 0$, we can often simplify the statement by substituting in finite expansions that we know.

• The finite expansions is very important for calculating limits with indeterminate forms ! It is enough just to note that if

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots \text{ thus } \lim_{x \to x_0} f(x) = a_0.$$

Applications

Example

$$\lim_{x \to 0} \frac{\ln(1+x) - \tan x + \frac{1}{2}\sin^2 x}{3x^2 \sin^2 x}$$

At 0

$$f(x) = \ln(1+x) - \tan x + \frac{1}{2}\sin^2 x = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + x^4\epsilon(x)\right) -$$

$$\left(x - \frac{x^3}{3} + x^4\epsilon(x)\right) + \frac{1}{2}\left(x - \frac{x^3}{6} + x^4\epsilon(x)\right)^2$$

and

$$g(x) = 3x^2 \sin^2 x = 3x^2(x + x\epsilon(x))^2 = 3x^4 + x^4\epsilon(x)$$

Also

$$\frac{f(x)}{g(x)} = \frac{-\frac{5}{12}x^4 + x^4\epsilon(x)}{3x^4 + x^4\epsilon(x)} = \boxed{-\frac{5}{36}}$$

Applications

Remark

Note that : by calculating the finite expansions at a lower order (2 for example), we would not have been able to conclude, because we would have obtained $\dfrac{f(x)}{g(x)} = \dfrac{x^2\epsilon(x)}{x^2\epsilon(x)}$, which does not remove the indeterminacy.

Generally, we calculate the finite expansion at the lowest possible order, and if this is not enough, we increase the order (and therefore the precision of the approximation).

Applications

Remark

∘ We know that if : $\lim_{x\to+\infty} f(x) = +\infty$ and $\lim_{x\to+\infty} \dfrac{f(x)}{x} = a$, then a represents the slope of the asymptotic straight line.
∘ Thus, the straight line with equation : $y = ax$, is an **oblique asymptote.**
∘ More generally, if : $\lim_{x\to+\infty}(f(x) - ax) = b$, then the graph of f admits the straight line $y = ax + b$ as an asymptote at infinity. Moreover :

- If $\lim_{x\to+\infty}(f(x) - y) = 0^+$, we said that the graph of f is above of the straight line asymptote.
- If $\lim_{x\to+\infty}(f(x) - y) = 0^-$, the graph of f been bellow of the straight line asymptote.

Applications

General remark

In general, if the function f has a finite expansion an a neighborhood of ∞ :

$$f(x) = \alpha x + \beta + \frac{\gamma}{x^p} + x^p \epsilon(\frac{1}{x}).$$

We obtained immediately equation of straight line asymptote $y = \alpha x + \beta$ and a sign of γ indicates the relative position towards of the graph of f.

Example

Expand the function $f : x \mapsto f(x) = \sqrt{1 + \frac{1}{x} + \frac{1}{x^2}}$ an a neighborhood of ∞ of order $n = 2$. Deduce the relative position of straight lines asymptotes towards the graph of the function $g : x \mapsto \sqrt{x^2 + x + 1}$.

Applications

We have $t = \frac{1}{x}$:

$$f(x) = \sqrt{1 + \frac{1}{x} + \frac{1}{x^2}} = \sqrt{t^2 + t + 1}.$$

And by setting $u = t^2 + t$ we get :

$$\sqrt{t^2 + t + 1} = \sqrt{1 + u} = 1 + \frac{u}{2} - \frac{3}{8}u^2 + u^2\epsilon(u)$$

$$= 1 + \frac{t^2 + t}{2} - \frac{3}{8}(t^2 + t)^2 + t^2\epsilon(t) = 1 + \frac{t}{2} + \frac{1}{8}t^2 + t^2\epsilon(t).$$

And from it :

$$f(x) = 1 + \frac{1}{2x} + \frac{1}{8x^2} + \frac{1}{x^2}\epsilon(\frac{1}{x}).$$

Applications

Therefore

$$g(x) = |x| \left(1 + \frac{1}{2x} + \frac{1}{8x^2} + \frac{1}{x^2}\epsilon(\frac{1}{x})\right).$$

That is :

$$g(x) = \begin{cases} x + \frac{1}{2} + \frac{1}{8x} + \frac{1}{x}\epsilon\left(\frac{1}{x}\right) & x \to +\infty \\ -x - \frac{1}{2} - \frac{1}{8x} - \frac{1}{x}\epsilon\left(\frac{1}{x}\right) & x \to -\infty \end{cases}$$

From this, we deduce that the straight lines with equations $y = x + \frac{1}{2}$ and $y = -x - \frac{1}{2}$ are oblique asymptotes of the graph of g at neighborhood $+\infty$ and $-\infty$ successively. We notice that the graph of g is above its asymptotes, because the term $\frac{1}{8x}$ at neighborhood $+\infty$ and the term $-\frac{1}{8x}$ at neighborhood $-\infty$ are positives.

Applications

Exercise

Expand the function $f : x \mapsto f(x) = \sqrt{\dfrac{x}{x-1}}$ an a neighborhood of ∞ of order $n = 2$. Deduce the relative position of straight lines asymptotes towards the graph of the function $g : x \mapsto \sqrt{\dfrac{x^3}{x-1}}$.

Solution

We have $t = \frac{1}{x}$:

$$\sqrt{\frac{1}{1-t}} = 1 + \frac{1}{2}t + \frac{3}{8}t^2 + t^2\epsilon(t)$$

$$= 1 + \frac{1}{2x} + \frac{3}{8x^2} + \frac{1}{x^2}\epsilon(\frac{1}{x})$$

Thus

$$g(x) = \begin{cases} x + \frac{1}{2} + \frac{3}{8x} + \frac{1}{x}\epsilon(\frac{1}{x}) & x \to +\infty \\ -x - \frac{1}{2} - \frac{3}{8x} - \frac{1}{x}\epsilon(\frac{1}{x}) & x \to -\infty \end{cases}$$

We deduce, the same of previous example.

End Chapter 1