Sample Correction for Series N°03.
Algebra 2

Exercise 1 :

1-**a**

1. Compute $-2A$:

$$-2A = -2 \cdot \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} -4 & -10 \\ -2 & -6 \end{pmatrix}$$

2. Compute A^2 :

$$A^2 = A \cdot A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 9 & 25 \\ 5 & 14 \end{pmatrix}$$

3. Compute B^2 :

$$B^2 = B \cdot B = \begin{pmatrix} 0 & 4 \\ 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 4 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 8 \\ 4 & 12 \end{pmatrix}$$

4. Compute $B^2 - 2A$:

$$B^2 - 2A = B^2 + (-2A) = \begin{pmatrix} 8 & 8 \\ 4 & 12 \end{pmatrix} + \begin{pmatrix} -4 & -10 \\ -2 & -6 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 2 & 6 \end{pmatrix}$$

5. Compute AC :

$$AC = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} (2 \cdot 3 + 5 \cdot -1) & (2 \cdot -5 + 5 \cdot 2) \\ (1 \cdot 3 + 3 \cdot -1) & (1 \cdot -5 + 3 \cdot 2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

6. Compute CA :

$$CA = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

7. Compute CD :

$$CD = C \cdot D = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} -1 & 3 & 0 \\ -4 & 5 & -1 \end{pmatrix} = \begin{pmatrix} 17 & -16 & 5 \\ -7 & 7 & -2 \end{pmatrix}$$

8. Compute DC :

$$DC = D \cdot C = \begin{pmatrix} -1 & 3 & 0 \\ -4 & 5 & -1 \end{pmatrix} \cdot \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$$

Since D has dimensions 2×3 and C has dimensions 2×2, the multiplication DC is **not defined** due to incompatible dimensions.

9. Compute $C - E$:

$$C - E = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Since C has dimensions 2×2 and E has dimensions 2×1, the subtraction $C - E$ is **not defined** due to incompatible dimensions.

10. Compute EB :

$$EB = E \cdot B = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 4 \\ 2 & 2 \end{pmatrix}$$

Since E has dimensions 2×1 and B has dimensions 2×2, the multiplication EB is **not defined** due to incompatible dimensions.

11. Compute BE :

$$BE = B \cdot E = \begin{pmatrix} 0 & 4 \\ 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$$

12. Compute FG :

$$FG = F \cdot G = \begin{pmatrix} -2 & 2 & 1 \\ 2 & 0 & -2 \\ 1 & 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} -2 & 1 & 2 \\ 3 & 1 & 5 \\ 4 & -6 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} (-2 \cdot -2 + 2 \cdot 3 + 1 \cdot 4) & (-2 \cdot 1 + 2 \cdot 1 + 1 \cdot -6) & (-2 \cdot 2 + 2 \cdot 5 + 1 \cdot -3) \\ (2 \cdot -2 + 0 \cdot 3 + -2 \cdot 4) & (2 \cdot 1 + 0 \cdot 1 + -2 \cdot -6) & (2 \cdot 2 + 0 \cdot 5 + -2 \cdot -3) \\ (1 \cdot -2 + 3 \cdot 3 + 2 \cdot 4) & (1 \cdot 1 + 3 \cdot 1 + 2 \cdot -6) & (1 \cdot 2 + 3 \cdot 5 + 2 \cdot -3) \end{pmatrix}$$

$$= \begin{pmatrix} 14 & -6 & 3 \\ -12 & 14 & 10 \\ 15 & -8 & 11 \end{pmatrix}$$

1-b

(a) Compute A^T :

$$A^T = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$$

(b) Compute B^T :

$$B^T = \begin{pmatrix} 0 & 2 \\ 4 & 2 \end{pmatrix}$$

(c) Compute D^T :

$$D^T = \begin{pmatrix} -1 & -4 \\ 3 & 5 \\ 0 & -1 \end{pmatrix}$$

(d) Compute $\mathrm{Tr}(A)$:

$$\mathrm{Tr}(A) = 2 + 3 = 5$$

(e) Compute $\mathrm{Tr}(B)$:

$$\mathrm{Tr}(B) = 0 + 2 = 2$$

(f) Compute $\mathrm{Tr}(D)$:

$$\mathrm{Tr}(D) \text{ is not defined since } D \text{ is not a square matrix.}$$

(g) Compute $\mathrm{Tr}(A + B)$:

$$\mathrm{Tr}(A + B) = \mathrm{Tr}(A) + \mathrm{Tr}(B) = 5 + 2 = 7$$

1-c

1. Compute $\det(A)$:

$$A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}, \quad \det(A) = (2 \cdot 3) - (5 \cdot 1) = 6 - 5 = 1$$

2. Compute $\det(A^T)$:

$$A^T = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}, \quad \det(A^T) = \det(A) = 1$$

3. Compute $\det(B)$:

$$B = \begin{pmatrix} 0 & 4 \\ 2 & 2 \end{pmatrix}$$

$$\det(B) = (0 \times 2) - (4 \times 2) = 0 - 8 = -8.$$

4. Compute $\det(A + B)$:
   First, compute $A + B$:

$$A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 4 \\ 2 & 2 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 2+0 & 5+4 \\ 1+2 & 3+2 \end{pmatrix} = \begin{pmatrix} 2 & 9 \\ 3 & 5 \end{pmatrix}$$

   Now, compute the determinant :

$$\det(A + B) = (2 \times 5) - (9 \times 3) = 10 - 27 = -17.$$

5. Compute $\det(AB)$:

   The determinant of a product of matrices satisfies :

$$\det(AB) = \det(A) \cdot \det(B).$$

   Using the computed values :

$$\det(A) = 1, \quad \det(B) = -8.$$

$$\det(AB) = 1 \times (-8) = -8.$$

6. Compute $\det(D)$:

   Matrix D is :

$$D = \begin{pmatrix} -1 & 3 & 0 \\ -4 & 5 & -1 \end{pmatrix}$$

   Since D is a 2×3 matrix, it is **not square**, and its determinant is **not defined**.

7. Compute $\det(F)$:

$$F = \begin{pmatrix} -2 & 2 & 1 \\ 2 & 0 & -2 \\ 1 & 3 & 2 \end{pmatrix}$$

   Expand along the first row :

$$\det(F) = (-1)^{1+1}(-2)\cdot\det\begin{pmatrix} 0 & -2 \\ 3 & 2 \end{pmatrix} + (-1)^{1+2}(2)\cdot\det\begin{pmatrix} 2 & -2 \\ 1 & 2 \end{pmatrix} + (-1)^{1+3}(1)\cdot\det\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$$

   Compute each cofactor :

$$(-1)^{1+1}(-2) = +(-2) = -2, \quad (-1)^{1+2}(2) = -(2) = -2, \quad (-1)^{1+3}(1) = +(1) = +1$$

   Compute each minor :

$$\det\begin{pmatrix} 0 & -2 \\ 3 & 2 \end{pmatrix} = (0 \cdot 2) - (-2 \cdot 3) = 6$$

$$\det\begin{pmatrix} 2 & -2 \\ 1 & 2 \end{pmatrix} = (2 \cdot 2) - (-2 \cdot 1) = 4 + 2 = 6$$

$$\det\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} = (2 \cdot 3) - (0 \cdot 1) = 6$$

   Substitute :

$$\det(F) = (-2)(6) + (-2)(6) + (1)(6) = -12 - 12 + 6 = -18$$

8. Compute $\det(G)$:

$$G = \begin{pmatrix} -2 & 1 & 2 \\ 3 & 1 & 5 \\ 4 & -6 & -3 \end{pmatrix}$$

Expand along the first row :

$$\det(G) = (-1)^{1+1}(-2)\cdot\det\begin{pmatrix} 1 & 5 \\ -6 & -3 \end{pmatrix} + (-1)^{1+2}(1)\cdot\det\begin{pmatrix} 3 & 5 \\ 4 & -3 \end{pmatrix} + (-1)^{1+3}(2)\cdot\det\begin{pmatrix} 3 & 1 \\ 4 & -6 \end{pmatrix}$$

Compute each cofactor :

$$(-1)^{1+1}(-2) = +(-2) = -2, \quad (-1)^{1+2}(1) = -(1) = -1, \quad (-1)^{1+3}(2) = +(2) = +2$$

Compute each minor :

$$\det\begin{pmatrix} 1 & 5 \\ -6 & -3 \end{pmatrix} = (1 \cdot -3) - (5 \cdot -6) = -3 + 30 = 27$$

$$\det\begin{pmatrix} 3 & 5 \\ 4 & -3 \end{pmatrix} = (3 \cdot -3) - (5 \cdot 4) = -9 - 20 = -29$$

$$\det\begin{pmatrix} 3 & 1 \\ 4 & -6 \end{pmatrix} = (3 \cdot -6) - (1 \cdot 4) = -18 - 4 = -22$$

Substitute :

$$\det(G) = (-2)(27) + (-1)(-29) + (2)(-22) = -54 + 29 - 44 = -69$$

2- **Method 1** : Using the Definition of an Inverse matrix

- A matrix A is invertible if there exists a matrix B such that :

$$A \cdot B = B \cdot A = I,$$

where I is the identity matrix.

From the previous question, we have :

$$A \cdot C = C \cdot A = I,$$

where I is the identity matrix. Therefore, A is invertible, and its inverse is C (the same reasoning applies to C).

Method 2 : Using the Determinant

- A matrix A is invertible if and only if :

$$\det(A) \neq 0.$$

Since :

$$\det(A) = 1,$$

we conclude that A is invertible. The same for C.

3-

1. Verify that F is invertible :

   The determinant of F is computed as :

$$\det(F) = (-1)^{1+1}(-2) \cdot \det\begin{pmatrix} 0 & -2 \\ 3 & 2 \end{pmatrix} + (-1)^{1+2}(2) \cdot \det\begin{pmatrix} 2 & -2 \\ 1 & 2 \end{pmatrix} + (-1)^{1+3}(1) \cdot \det\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$$

   Compute each cofactor :

$$(-1)^{1+1}(-2) = -2, \quad (-1)^{1+2}(2) = -2, \quad (-1)^{1+3}(1) = 1$$

   Compute minors :

$$\det\begin{pmatrix} 0 & -2 \\ 3 & 2 \end{pmatrix} = (0 \cdot 2) - (-2 \cdot 3) = 6$$

$$\det\begin{pmatrix} 2 & -2 \\ 1 & 2 \end{pmatrix} = (2 \cdot 2) - (-2 \cdot 1) = 4 + 2 = 6$$

$$\det\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} = (2 \cdot 3) - (0 \cdot 1) = 6$$

   Substitute :

$$\det(F) = (-2)(6) + (-2)(6) + (1)(6) = -12 - 12 + 6 = -18$$

   Since $\det(F) \neq 0$, F is invertible.

2. Calculate the cofactor matrix $\mathrm{Cof}(F)$:

   Use the formula $\mathrm{Cof}(F) = ((-1)^{i+j} \cdot M_{ij})$, where M_{ij} is the determinant of the minor matrix obtained by deleting row i and column j. The cofactor matrix $\mathrm{Cof}(F)$ is obtained by computing each cofactor $C_{ij} = (-1)^{i+j} \cdot M_{ij}$, where M_{ij} is the determinant of the minor matrix obtained by removing the i-th row and j-th column from F.

(a) Compute the cofactors for Row 1 :

$$C_{11} = (-1)^{1+1} \cdot \det \begin{pmatrix} 0 & -2 \\ 3 & 2 \end{pmatrix} = (1) \cdot [(0 \cdot 2) - (-2 \cdot 3)] = 6$$

$$C_{12} = (-1)^{1+2} \cdot \det \begin{pmatrix} 2 & -2 \\ 1 & 2 \end{pmatrix} = (-1) \cdot [(2 \cdot 2) - (-2 \cdot 1)] = -6$$

$$C_{13} = (-1)^{1+3} \cdot \det \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} = (1) \cdot [(2 \cdot 3) - (0 \cdot 1)] = 6$$

(b) Compute the cofactors for Row 2 :

$$C_{21} = (-1)^{2+1} \cdot \det \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} = (-1) \cdot [(2 \cdot 2) - (1 \cdot 3)] = -1$$

$$C_{22} = (-1)^{2+2} \cdot \det \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} = (1) \cdot [(-2 \cdot 2) - (1 \cdot 1)] = -5$$

$$C_{23} = (-1)^{2+3} \cdot \det \begin{pmatrix} -2 & 2 \\ 1 & 3 \end{pmatrix} = (-1) \cdot [(-2 \cdot 3) - (2 \cdot 1)] = 8$$

(c) Compute the cofactors for Row 3 :

$$C_{31} = (-1)^{3+1} \cdot \det \begin{pmatrix} 2 & 1 \\ 0 & -2 \end{pmatrix} = (1) \cdot [(2 \cdot -2) - (1 \cdot 0)] = -4$$

$$C_{32} = (-1)^{3+2} \cdot \det \begin{pmatrix} -2 & 1 \\ 2 & -2 \end{pmatrix} = (-1) \cdot [(-2 \cdot -2) - (1 \cdot 2)] = -2$$

$$C_{33} = (-1)^{3+3} \cdot \det \begin{pmatrix} -2 & 2 \\ 2 & 0 \end{pmatrix} = (1) \cdot [(-2 \cdot 0) - (2 \cdot 2)] = -4$$

The cofactor matrix $\mathrm{Cof}(F)$ is :

$$\mathrm{Cof}(F) = \begin{pmatrix} 6 & -6 & 6 \\ -1 & -5 & 8 \\ -4 & -2 & -4 \end{pmatrix}$$

3. Deduce the inverse matrix F^{-1} :

The formula for F^{-1} is :

$$F^{-1} = \frac{1}{\det(F)} \cdot \mathrm{Cof}(F)^T$$

Substitute $\det(F)$ and transpose the cofactor matrix to compute F^{-1}.

$$F^{-1} = \frac{1}{-18} \cdot \begin{pmatrix} 6 & -1 & -4 \\ -6 & -5 & -2 \\ 6 & 8 & -4 \end{pmatrix}$$

$$F^{-1} = \begin{pmatrix} \frac{6}{-18} & \frac{-1}{-18} & \frac{-4}{-18} \\ \frac{-6}{-18} & \frac{-5}{-18} & \frac{-2}{-18} \\ \frac{6}{-18} & \frac{8}{-18} & \frac{-4}{-18} \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{1}{18} & \frac{2}{9} \\ \frac{1}{3} & \frac{5}{18} & \frac{1}{9} \\ -\frac{1}{3} & -\frac{4}{9} & \frac{2}{9} \end{pmatrix}$$

4. Repeat the same for G :

$$G^{-1} = \frac{1}{\det(G)} \cdot \mathrm{Cof}(G)^T$$

Exercise 02 :

We start by calculating the first terms of A^n to try to guess the formula. We have :

$$A^2 = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}.$$

We then prove by induction for $n \geq 1$ that :

$$A^n = \begin{bmatrix} 2^{n-1} & -2^{n-1} \\ -2^{n-1} & 2^{n-1} \end{bmatrix}.$$

The proof by induction is very simple and is based on the fact that :

$$A^n = 2^{n-1}.A, \text{then, } A^{n+1} = A^n.A = 2^{n-1}.A^2 = 2^{n-1}.2.A = 2^n.A.$$

Similarly, for B, we have :

$$B^2 = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}, \quad B^3 = \begin{bmatrix} 1 & 7 \\ 0 & 8 \end{bmatrix}.$$

We then prove by induction on n that :

$$B^n = \begin{bmatrix} 1 & 2^n - 1 \\ 0 & 2^n \end{bmatrix}.$$

Exercise 03 :

1. **Determine the matrix B such that $A = I_2 + 4B$:**

   Solve for B :

$$B = \frac{1}{4}(A - I_2)$$

   Compute $A - I_2$:

$$A - I_2 = \begin{pmatrix} 5-1 & -4-0 \\ 4-0 & -3-1 \end{pmatrix} = \begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}$$

   Multiply by $\frac{1}{4}$:

$$B = \begin{pmatrix} \frac{4}{4} & \frac{-4}{4} \\ \frac{4}{4} & \frac{-4}{4} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

2. **Calculate the matrix $-A^2 + 2A - I_2$:**

   Compute A^2 :

$$A^2 = A \cdot A = \begin{pmatrix} 5 & -4 \\ 4 & -3 \end{pmatrix} \cdot \begin{pmatrix} 5 & -4 \\ 4 & -3 \end{pmatrix}$$

Compute each entry :

$$A^2 = \begin{pmatrix} (5 \cdot 5 + (-4) \cdot 4) & (5 \cdot (-4) + (-4) \cdot (-3)) \\ (4 \cdot 5 + (-3) \cdot 4) & (4 \cdot (-4) + (-3) \cdot (-3)) \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 25 - 16 & -20 + 12 \\ 20 - 12 & -16 + 9 \end{pmatrix} = \begin{pmatrix} 9 & -8 \\ 8 & -7 \end{pmatrix}$$

Compute $-A^2 + 2A - I_2$:

$$-A^2 = \begin{pmatrix} -9 & 8 \\ -8 & 7 \end{pmatrix}$$

$$2A = \begin{pmatrix} 10 & -8 \\ 8 & -6 \end{pmatrix}$$

$$-A^2 + 2A - I_2 = \begin{pmatrix} -9 + 10 - 1 & 8 - 8 - 0 \\ -8 + 8 - 0 & 7 - 6 - 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

3. **Determine whether A is invertible and calculate A^{-1} if it exists :**
   (a) Since $-A^2 + 2A - I_2 = 0$, we deduce :

   $$A^2 - 2A + I_2 = 0$$

   Rearranging :

   $$A(A - 2I_2) = I_2$$

   Thus, by the definition of the inverse matrix, we conclude :

   $$A^{-1} = -A + 2I_2.$$

   (b) **Compute A^{-1} :**

   $$A^{-1} = -A + 2I_2$$

   Compute $-A$:

   $$-A = \begin{pmatrix} -5 & 4 \\ -4 & 3 \end{pmatrix}$$

   Compute $2I_2$:

   $$2I_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

   Compute A^{-1} :

   $$A^{-1} = \begin{pmatrix} -5 + 2 & 4 + 0 \\ -4 + 0 & 3 + 2 \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ -4 & 5 \end{pmatrix}$$

4. **We show that** $\forall n \in \mathbb{N}^*, A^n = I_2 + 4nB$

   (a) **Base Case ($n = 1$) :**
   For $n = 1$, we check :

   $$A^1 = I_2 + 4B.$$

   Since $A = I_2 + 4B$, we conclude :

   $$A^1 = A = I_2 + 4B.$$

   Thus, the formula holds for $n = 1$.

   (b) **Inductive Hypothesis :**
   Assume that for some $n \geq 1$,

   $$A^n = I_2 + 4nB.$$

   We must prove that it holds for $n + 1$, i.e.,

   $$A^{n+1} = I_2 + 4(n+1)B.$$

   (c) **Inductive Step :**
   Using the inductive hypothesis :

   $$A^{n+1} = A^n \cdot A.$$

   Substituting $A^n = I_2 + 4nB$ and $A = I_2 + 4B$:

   $$A^{n+1} = (I_2 + 4nB) \cdot (I_2 + 4B).$$

   Expanding :

   $$A^{n+1} = I_2 \cdot I_2 + I_2 \cdot 4B + 4nB \cdot I_2 + 4nB \cdot 4B.$$

   Since $I_2 \cdot I_2 = I_2$ and $I_2 \cdot B = B$:

   $$A^{n+1} = I_2 + 4B + 4nB + 16nB^2.$$

   We can verify that $B^2 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, so $16nB^2 = 0$, giving :

   $$A^{n+1} = I_2 + 4B + 4nB.$$

   Rewriting :

   $$A^{n+1} = I_2 + 4(n+1)B.$$

   By the principle of mathematical induction, the statement :

   $$A^n = I_2 + 4nB$$

   holds for all $n \in \mathbb{N}^*$.