MOHAMED BOUDIAF UNIVERSITY-M'SILA
First year in Computer Science
DEPARTMENT OF COMPUTER SCIENCE
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Sample Correction for Series N°02.
Algebra 2

Exercise 01 :
We verify each function using these conditions.

1. $f_1 : \mathbb{R}^2 \to \mathbb{R}$

$$f_1(x_1, x_2) = x_1 + x_2$$

Additivity : For all $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$:

$$f_1((x_1, x_2) + (y_1, y_2)) = f_1(x_1 + y_1, x_2 + y_2)$$

$$= (x_1 + y_1) + (x_2 + y_2)$$

$$= f_1(x_1, x_2) + f_1(y_1, y_2)$$

Homogeneity : For all $\alpha \in \mathbb{R}$ and $(x_1, x_2) \in \mathbb{R}^2$:

$$f_1(\alpha(x_1, x_2)) = f_1(\alpha x_1, \alpha x_2)$$

$$= \alpha x_1 + \alpha x_2$$

$$= \alpha f_1(x_1, x_2)$$

Thus, f_1 is linear.

2. $f_2 : \mathbb{R}^2 \to \mathbb{R}^2$

$$f_2(x_1, x_2) = (x_1 - x_2, 0)$$

Additivity : For all $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$:

$$f_2((x_1, x_2) + (y_1, y_2)) = f_2(x_1 + y_1, x_2 + y_2)$$

$$= (x_1 + y_1 - x_2 - y_2, 0)$$

$$= (x_1 - x_2, 0) + (y_1 - y_2, 0) = f_2(x_1, x_2) + f_2(y_1, y_2)$$

Homogeneity : For all $\alpha \in \mathbb{R}$ and $(x_1, x_2) \in \mathbb{R}^2$:

$$f_2(\alpha(x_1, x_2)) = f_2(\alpha x_1, \alpha x_2)$$

$$= (\alpha x_1 - \alpha x_2, 0)$$

$$= \alpha(x_1 - x_2, 0) = \alpha f_2(x_1, x_2)$$

Thus, f_2 is linear.

3. $f_3 : \mathbb{R}^2 \to \mathbb{R}^2$

$$f_3(x_1, x_2) = (x_1 - x_2, -2x_1 + 2x_2)$$

For all $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$ and $\alpha, \beta \in \mathbb{R}$:

$$f_3(\alpha(x_1, x_2) + \beta(y_1, y_2)) = f_3(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2)$$

$$= (\alpha x_1 + \beta y_1 - \alpha x_2 - \beta y_2, -2\alpha x_1 - 2\beta y_1 + 2\alpha x_2 + 2\beta y_2)$$

$$= \alpha(x_1 - x_2, -2x_1 + 2x_2) + \beta(y_1 - y_2, -2y_1 + 2y_2)$$

$$= \alpha f_3(x_1, x_2) + \beta f_3(y_1, y_2)$$

Thus, f_3 is linear.

4. $f_4 : \mathbb{R}^3 \to \mathbb{R}^2$

$$f_4(x_1, x_2, x_3) = (-2x_1 + x_2 + x_3, x_1 - 2x_2 + x_3)$$

For all $(x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbb{R}^3$ and $\alpha, \beta \in \mathbb{R}$:

$$f_4(\alpha(x_1, x_2, x_3) + \beta(y_1, y_2, y_3)) = f_4(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3)$$

$$= (-2\alpha x_1 - 2\beta y_1 + \alpha x_2 + \beta y_2 + \alpha x_3 + \beta y_3, \alpha x_1 + \beta y_1 - 2\alpha x_2 - 2\beta y_2 + \alpha x_3 + \beta y_3)$$

$$= \alpha f_4(x_1, x_2, x_3) + \beta f_4(y_1, y_2, y_3)$$

Thus, f_4 is linear.

5. $f_5 : \mathbb{R}^3 \to \mathbb{R}^3$

$$f_5(x_1, x_2, x_3) = (-x_1 + x_2, x_1 - x_3, x_2)$$

For all $(x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbb{R}^3$ and $\alpha, \beta \in \mathbb{R}$:

$$f_5(\alpha(x_1, x_2, x_3) + \beta(y_1, y_2, y_3)) = f_5(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3)$$

$$= (-\alpha x_1 - \beta y_1 + \alpha x_2 + \beta y_2, \alpha x_1 + \beta y_1 - \alpha x_3 - \beta y_3, \alpha x_2 + \beta y_2)$$

$$= \alpha f_5(x_1, x_2, x_3) + \beta f_5(y_1, y_2, y_3)$$

Thus, f_5 is linear.

6. $f_6 : \mathbb{R}^3 \to \mathbb{R}^2$

$$f_6(x_1, x_2, x_3) = (x_1 + \sqrt{2}x_2 + x_3, 2x_1 + x_2 - 5x_3)$$

Additivity : For all $(x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbb{R}^3$:

$$f_6((x_1, x_2, x_3) + (y_1, y_2, y_3)) = f_6(x_1 + y_1, x_2 + y_2, x_3 + y_3)$$

$$= (x_1 + y_1 + \sqrt{2}(x_2 + y_2) + x_3 + y_3, 2(x_1 + y_1) + (x_2 + y_2) - 5(x_3 + y_3))$$

$$= (x_1 + \sqrt{2}x_2 + x_3, 2x_1 + x_2 - 5x_3) + (y_1 + \sqrt{2}y_2 + y_3, 2y_1 + y_2 - 5y_3)$$

$$= f_6(x_1, x_2, x_3) + f_6(y_1, y_2, y_3)$$

Homogeneity : For all $\alpha \in \mathbb{R}$ and $(x_1, x_2, x_3) \in \mathbb{R}^3$:

$$f_6(\alpha(x_1, x_2, x_3)) = f_6(\alpha x_1, \alpha x_2, \alpha x_3)$$

$$= (\alpha x_1 + \sqrt{2}\alpha x_2 + \alpha x_3, 2\alpha x_1 + \alpha x_2 - 5\alpha x_3)$$

$$= \alpha(x_1 + \sqrt{2}x_2 + x_3, 2x_1 + x_2 - 5x_3) = \alpha f_6(x_1, x_2, x_3)$$

Thus, f_6 is linear.

7. $f_7 : \mathbb{R} \to \mathbb{R}^2$

$$f_7(x_1) = (x_1, -x_1)$$

For all $x_1, y_1 \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$:

$$f_7(\alpha x_1 + \beta y_1) = (\alpha x_1 + \beta y_1, -(\alpha x_1 + \beta y_1))$$

$$= (\alpha x_1, -\alpha x_1) + (\beta y_1, -\beta y_1)$$

$$= \alpha f_7(x_1) + \beta f_7(y_1)$$

Thus, f_7 is linear.

Exercise 02 :

1. f is linear if and only if $\forall v = (x, y, z), u = (x_0, y_0, z_0) \in \mathbb{R}^3$, we have :

$$f(\lambda v + \mu u) = \lambda f(v) + \mu f(u).$$

We know :

$$\lambda v + \mu u = (\lambda x + \mu x_0, \lambda y + \mu y_0, \lambda z + \mu z_0).$$

Let :

$$X = \lambda x + \mu x_0, \quad Y = \lambda y + \mu y_0, \quad Z = \lambda z + \mu z_0.$$

Then :

$$f(\lambda v + \mu u) = f(X, Y, Z) = (Y + Z, X + Z, X)$$
$$= (\lambda(y + z) + \mu(y_0 + z_0), \lambda(x + z) + \mu(x_0 + z_0), \lambda x + \mu x_0)$$
$$= \lambda(y + z, x + z, x) + \mu(y_0 + z_0, x_0 + z_0, x_0)$$
$$= \lambda f(x, y, z) + \mu f(x_0, y_0, z_0) = \lambda f(v) + \mu f(u).$$

Hence, f is linear. Since $E = F = \mathbb{R}^3$, f is an endomorphism.

2. • $\ker(f) = \{u \in \mathbb{R}^3 \mid f(u) = 0_{\mathbb{R}^3}\}$, so :

$$f(u) = 0_{\mathbb{R}^3} \iff (y + z, x + z, x) = (0, 0, 0)$$
$$\iff \begin{cases} y + z = 0 \\ x + z = 0 \\ x = 0 \end{cases} \iff \begin{cases} y = 0 \\ z = 0 \\ x = 0 \end{cases}$$

Thus :

$$\ker(f) = \{(0, 0, 0)\} = \{0_{\mathbb{R}^3}\},$$

so f is injective.

- $\mathrm{Im}(f) = \{f(u) \mid u \in \mathbb{R}^3\}$ is a vector subspace of $\mathbb{R}^3$. We know :

$$\dim(\mathrm{Im}(f)) = \mathrm{rank}(f) = \dim(\mathbb{R}^3) - \dim(\ker(f)) = 3 - 0 = 3,$$

  so :

$$\mathrm{Im}(f) = \mathbb{R}^3.$$

  Therefore, f is surjective. Since f is both injective and surjective, it is bijective, and hence f is an automorphism.

3. Since f is bijective, f^{-1} exists with $f^{-1} : \mathbb{R}^3 \to \mathbb{R}^3$. To find the formula for f^{-1}, we use the equivalence :

$$f(x, y, z) = (x_0, y_0, z_0) \iff f^{-1}(x_0, y_0, z_0) = (x, y, z).$$

   Then :

$$f(x, y, z) = (x_0, y_0, z_0) \iff \begin{cases} y + z = x_0, \\ x + z = y_0, \\ x = z_0, \end{cases}$$

   which implies :

$$\begin{cases} y = x_0 - y_0 + z_0, \\ z = y_0 - z_0, \\ x = z_0. \end{cases}$$

   Thus :

$$f^{-1}(x_0, y_0, z_0) = (x, y, z) = (z_0, x_0 - y_0 + z_0, y_0 - z_0).$$

Exercise 03 :

1. **Kernel of f :**
   The kernel of f is defined as :

$$\ker(f) = \{u \in \mathbb{R}^3 \mid f(u) = 0_{\mathbb{R}^2}\}.$$

   Thus :

$$f(u) = 0_{\mathbb{R}^2} \iff f(x, y, z) = (0, 0).$$

   This is equivalent to :

$$\begin{cases} -2x + y + z = 0, \\ x - 2y + z = 0. \end{cases}$$

   Solving this system :

$$-2x + y + z = 0,$$
$$x - 2y + z = 0.$$

   Substituting and simplifying :

$$x = y = z.$$

   Therefore :

$$\ker(f) = \{u = (y, y, y) \mid y \in \mathbb{R}\} = \mathrm{Span}((1, 1, 1)).$$

Since $(1,1,1)$ is a unique nonzero vector, it forms a basis, and :

$$\dim(\ker(f)) = 1.$$

We know :

$$\operatorname{rank}(f) = \dim(\mathbb{R}^3) - \dim(\ker(f)) = 3 - 1 = 2.$$

Since $\operatorname{Im}(f)$ is a subspace of $\mathbb{R}^2$, we conclude :

$$\operatorname{Im}(f) = \mathbb{R}^2.$$

2. **Image of f :**
   Let $v \in \operatorname{Im}(f)$. Then :

   $$v = (-2x + y + z, x - 2y + z) = x(-2,1) + y(1,-2) + z(1,1).$$

   This means :

   $$\operatorname{Im}(f) = \operatorname{Span}(v_1, v_2, v_3), \quad \text{where } v_1 = (-2,1), v_2 = (1,-2), v_3 = (1,1).$$

   Since $\operatorname{rank}(f) = 2$, we can extract a basis from $\{v_1, v_2, v_3\}$. For example :

   $$\{v_1 = (-2,1), v_2 = (1,-2)\}, \quad \text{if they are linearly independent.}$$

   To verify linear independence :

   $$\alpha v_1 + \beta v_2 = (0,0) \iff \begin{cases} -2\alpha + \beta = 0, \\ \alpha - 2\beta = 0. \end{cases}$$

   Solving :

   $$\alpha = \beta = 0.$$

   Thus, the set $\{v_1, v_2\}$ is a basis for $\operatorname{Im}(f)$.

3. **Isomorphism :**
   From the first part, we have $\ker(f) \neq \{0_{\mathbb{R}^3}\}$. Therefore, f is not injective, and hence not bijective. Consequently, f is not an isomorphism.

Exercise 04 :

1. **Linearity of g :**
   We show that g is linear using the same method as in previous exercises.

2. **Kernel of g :**
   Calculate $\ker(g)$ to determine if g is injective :

   Let $u = \begin{pmatrix} x \\ y \end{pmatrix}$. Then :

   $$g(u) = g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y \\ -2x + 2y \end{pmatrix}.$$

   Setting $g(u) = 0_{\mathbb{R}^2}$, we have :

$$\begin{pmatrix} x-y \\ -2x+2y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives the following system of equations :

$$\begin{aligned} x - y &= 0, \\ -2x + 2y &= 0. \end{aligned}$$

$$\implies x = y.$$

Thus,

$$\ker(g) = \text{Span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \neq \{0\}.$$

Hence, g is not injective. Moreover :

$$\text{rank}(g) = \dim(\mathbb{R}^2) - \dim(\ker(g)) = 2 - 1 = 1.$$

This implies :

$$\text{Im}(g) \neq \mathbb{R}^2, \quad \text{i.e., } g \text{ is not surjective.}$$

3. **Image of g :**
From the previous question,

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ is a basis of } \ker(g).$$

Next, we calculate :

$$\text{Im}(g) = \{g(x,y) \mid (x,y) \in \mathbb{R}^2\} = \begin{pmatrix} x-y \\ -2x+2y \end{pmatrix}, \quad (x,y) \in \mathbb{R}^2.$$

This can be expressed as :

$$\text{Im}(g) = \left\{ x \underbrace{\begin{pmatrix} 1 \\ -2 \end{pmatrix}}_{v_1} + y \underbrace{\begin{pmatrix} -1 \\ 2 \end{pmatrix}}_{v_2}, \quad (x,y) \in \mathbb{R}^2 \right\}.$$

Here, $v_2 = -v_1$, so we can choose : v_1 as the basis of $\text{Im}(g)$.