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Sample Correction for Series N°01.
Algebra 2

Exercise 01 :

A- Let $E = \mathbb{R}^* \times \mathbb{R} = \{(a, b) : a, b \in \mathbb{R}, a \neq 0\}$ and $+$ be defined on E by

$$(a, b) + (c, d) = (ac, b + d)$$

1. $\star$ a is **vector addition** (Internal composition law) on E : For any $(a, b), (c, d) \in E$,

$$(a, b) + (c, d) = (ac, b + d)$$

   Since $a, c \neq 0$, $ac \neq 0$, and $b + d \in \mathbb{R}$, $(ac, b + d) \in E$. Hence, E is closed under vector addition $+$.

2. Is $(E, +)$ a commutative group ?

   To show that $(E, +)$ is a commutative group, we need to verify that it satisfies the group axioms : associativity, identity element, inverse element, and commutativity.

   - We begin by **commutativity** : For any $(a, b), (c, d) \in E$,

   $$(a, b) + (c, d) = (ac, b + d) = (ca, d + b) = (c, d) + (a, b)$$

     Hence, $+$ is commutative.

   - **Associativity** : For any $(a, b), (c, d), (e, f) \in E$,

   $$\begin{aligned} [(a, b) + (c, d)] + (e, f) &= (ac, b + d) + (e, f) \\ &= ((ac)e, (b + d) + f) \\ &= (a(ce), b + (d + f)) \\ &= (a, b) + [(c, d) + (e, f)] \end{aligned}$$

     Therefore, $+$ is associative.

   - **Identity element** : Assume that $(e_1, e_2) \in E$ is an identity element, it must satisfy $(a, b) + (e_1, e_2) = (a, b)$ for all $(a, b) \in E$: We have

   $$(a, b) + (e_1, e_2) = (a, b) \Longrightarrow (a \cdot e_1, b + e_2) = (a, b) \Longrightarrow (e_1, e_2) = (1, 0)$$

     So, $(e_1, e_2) = (1, 0)$ is the identity element.

   - **Inverse element** : For any $(a, b) \in E$, Let $(a', b') \in E$ such that

   $$(a, b) + (a', b') = (1, 0) \Longrightarrow (a \cdot a', b + b') = (1, 0) \Longrightarrow (a', b') = (a^{-1}, -b)$$

     Thus, each $(a, b) \in E$ has an inverse $(a^{-1}, -b) \in E$.

Since $(E,+)$ satisfies all group axioms, it is a commutative group.

B- Let an operation . be defined on E by :

$$\forall (a,b)\in E, \forall \lambda \in \mathbb{R} : \lambda.(a,b) = (a^{\lambda}, \lambda b)$$

1. $\perp$ is a **scalar multiplication** (External composition law) on E : For any $\lambda \in \mathbb{R}$ and $(a,b) \in E$,

$$\lambda.(a,b) = (a^{\lambda}, \lambda b) \in E$$

   Thus, E is closed under ..

2. Show that $(E,+,.)$ is a vector space over $\mathbb{R}$.

In addition of $(E,+)$ is a commutative group. To prove that $(E,+,.)$ is a vector space over $\mathbb{R}$, we need to verify the other fourth vector space axioms . Let $(a,b),(c,d) \in E$ and $\lambda, \mu \in \mathbb{R}$:

1. $\forall \lambda \in \mathbb{R}$, $\forall (a,b),(c,d) \in E$,

$$\begin{aligned}\lambda.((a,b)+(c,d)) &= \lambda.(ac, b+d)\\ &= ((ac)^{\lambda}, \lambda(b+d))\\ &= (a^{\lambda}c^{\lambda}, \lambda.b+\lambda.d)\\ &= (a^{\lambda}, \lambda.b)+(c^{\lambda}, \lambda.d)\\ &= (\lambda.(a,b)) + (\lambda.(c,d))\end{aligned}$$

   Therefore, the distributive property holds.

2. **Associativity of scalar multiplication** : For any $\lambda, \mu \in \mathbb{R}$ and $(a,b) \in E$,

$$\lambda.(\mu.(a,b)) = \lambda.(a^{\mu}, \mu b) = (a^{\lambda\mu}, \lambda.\mu.b) = (a^{(\lambda\mu)}, (\lambda.\mu).b) = (\lambda.\mu).(a,b)$$

   Thus, scalar multiplication is associative ($\lambda\mu = \lambda.\mu$).

3. **Distributivity of scalar addition** : For any $\lambda, \mu \in \mathbb{R}$ and $(a,b) \in E$,

$$(\lambda+\mu).(a,b) = (a^{\lambda+\mu}, (\lambda+\mu).b)$$

   and

$$(\lambda.(a,b)) + (\mu.(a,b)) = (a^{\lambda}, \lambda.b) + (a^{\mu}, \mu.b) = (a^{\lambda}a^{\mu}, \lambda.b+\mu.b) = (a^{\lambda+\mu}, (\lambda+\mu).b)$$

   Thus, scalar addition is distributive

4. **Identity element of scalar multiplication** : For any $(a,b) \in E$,

$$1.(a,b) = (a^1, 1b) = (a,b)$$

   Thus, the identity element for scalar multiplication is 1.

Since $(E,+,.)$ satisfies all the axioms of a vector space over $\mathbb{R}$, it is indeed an $\mathbb{R}$-vector space.

Exercise 2 :

- $E_1 = \{(x, y, z) \in \mathbb{R}^3 \mid x + y = 0\}$

  i) The zero vector (neutral element) of $\mathbb{R}^2$, $0_{\mathbb{R}^2} = (0, 0)$, belongs to E_1 since $x + y = 0 + 0 = 0$.

  ii) E_1 is closed under **vector addition** : let $u = (x, y) \in E_1$, $v = (x', y') \in E_1$, i.e., $x + y = 0$ and $x' + y' = 0$. We have :

  $$u+v = (x, y)+(x', y') = (x+x', y+y') = (x+x')+(y+y') = (x+y)+(x'+y') = 0+0 = 0$$

  iii) E_1 is closed under **scalar multiplication** : let $\lambda \in \mathbb{R}$ and let $u = (x, y) \in E_1$ i.e., $x + y = 0$. We have :

  $$\lambda.u = \lambda.(x, y) = \lambda.(x + y) = \lambda.0 = 0$$

  Thus, E_1 is a vector subspace of $\mathbb{R}^2$.

- $E_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x + y = z\}$.

  i) The zero vector (neutral element) of $\mathbb{R}^3$, $0_{\mathbb{R}^3} = (0, 0)$, belongs to E_2 since $x + y = z \Leftrightarrow 0 + 0 = 0$.

  ii) Let $\lambda, \mu \in \mathbb{R}$ and let $u = (x, y, z) \in E_2$, $v = (x', y', z') \in E_2$, i.e., $x + y = z$ and $x' + y' = z'$. We have :

  $$\lambda u+\mu v = \lambda(x, y, z)+\mu(x', y', z') = (\lambda x+\mu x', \lambda y+\mu y', \lambda z+\mu z') = \lambda(x+y)+\mu(x'+y') = \lambda z+\mu z'$$

  Thus, E_2 is a vector subspace of $\mathbb{R}^3$.

- $E_3 = \{(x, y, z) \in \mathbb{R}^3 \mid x - y + 5z = a, a \in \mathbb{R}\}$. We distinguish two cases :

  - **Case 1 :** If $a \neq 0$, $0_{\mathbb{R}^3} \notin E_3$ since $x - y + 5z = 0 - 0 + 5 \cdot 0 = 0 \neq a$, and thus E_3 is not a vector subspace.
  - **Case 2 :** If $a = 0$,

    i) The neutral element of $\mathbb{R}^3$, $0_{\mathbb{R}^3} = (0, 0, 0)$, belongs to E_3 since $x-y+5z = 0 = a$.

    ii) Let $\lambda, \mu \in \mathbb{R}$ and let $u = (x, y, z) \in E_3$, $v = (x', y', z') \in E_3$, i.e., $x - y + 5z = 0$ and $x' - y' + 5z' = 0$. We have :

    $$(\lambda x + \mu x') - (\lambda y + \mu y') + 5(\lambda z + \mu z') = \lambda(x - y + 5z) + \mu(x' - y' + 5z') = 0 = a$$

    Thus, $\lambda u + \mu v \in E_3$, and consequently, E_3 is a vector subspace of $\mathbb{R}^3$.

- $E_4 = \{(x, y, z) \in \mathbb{R}^3 \mid z = 1\}$ is not a vector subspace because the zero vector $0_{\mathbb{R}^3} \notin E_4$ since $z = 0 \neq 1$.

- $E_5 = \{(x, y, z, t) \in \mathbb{R}^4 \mid x + y + z + 2t = 0 \text{ and } x + y = 0\}$ is a vector subspace of $\mathbb{R}^4$. Indeed,

i) The neutral element of $\mathbb{R}^4$, $0_{\mathbb{R}^4} = (0, 0, 0, 0)$, belongs to E_5 since $x + y + z + 2t = 0$ and $x + y = 0$.

ii) Let $\lambda, \mu \in \mathbb{R}$ and let $u = (x, y, z, t) \in E_5$, $v = (x', y', z', t') \in E_5$, i.e., $x + y + z + 2t = 0$ and $x + y = 0$, and $x' + y' + z' + 2t' = 0$ and $x' + y' = 0$. We have :

$$\lambda u + \mu v = \lambda(x, y, z, t) + \mu(x', y', z', t') = (\lambda x + \mu x', \lambda y + \mu y', \lambda z + \mu z', \lambda t + \mu t')$$

Moreover,

$$(\lambda x+\mu x')+(\lambda y+\mu y')+(\lambda z+\mu z')+2(\lambda t+\mu t') = \lambda(x+y+z+2t)+\mu(x'+y'+z'+2t') = 0$$

and

$$(\lambda x + \mu x') + (\lambda y + \mu y') = \lambda(x + y) + \mu(x' + y') = 0$$

Thus, $\lambda u + \mu v \in E_5$ (i.e, E_5 is closed under vector addition and scalar multiplication). So, E_5 is vector subspace.

- $E_6 = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$

  i) The zero vector (The neutral element) $(0, 0) \in E_6$ since $0 \cdot 0 = 0$.

  ii) Let $(x_1, y_1), (x_2, y_2) \in E_6$. Then $x_1y_1 = 0$ and $x_2y_2 = 0$. We need to check if $(x_1 + x_2, y_1 + y_2) \in E_6$:

  $$(x_1 + x_2)(y_1 + y_2) = x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2$$

  Since $x_1y_1 = 0$ and $x_2y_2 = 0$, the sum is not necessarily zero. Thus, E_6 is not closed under addition.

  Since E_6 is not closed under vector addition, E_6 is not a vector subspace.

- $E_7 = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$

  i) The zero vector (The neutral element) $(0, 0) \in E_7$ since $0 = 0^2$.

  ii)

  1. Closed under vector addition : Let $(x_1, y_1), (x_2, y_2) \in E_7$ and $\lambda, \mu \in \mathbb{R}$. Then $y_1 = \lambda x_1^2$ and $y_2 = \mu x_2^2$. We need to check if $(\lambda x_1 + \mu x_2, \lambda y_1 + \mu y_2)) \in E_7$:

     $$\lambda y_1 + \mu y_2 = \lambda x_1^2 + \mu x_2^2 \neq (\lambda x_1 + \mu x_2)^2$$

     Thus, E_7 is not closed under addition. Then, it is not a vector subspace.

  2. $E_8 = P \in \mathbb{R}[X] \mid \deg(P) \leq 2$

     – Zero Vector : The zero polynomial $P(X) = 0$ is in E_8 since $\deg(0) \leq 2$.

     – Closed under vector addition : Let $P_1(X), P_2(X) \in E_8$. Then $\deg(P_1) \leq 2$ and $\deg(P_2) \leq 2$. We need to check if $P_1(X) + P_2(X) \in E_8$:

     $$\deg(P_1 + P_2) \leq \max(\deg(P_1), \deg(P_2)) \leq 2$$

     Thus, E_8 is closed under vector addition.

– Closed under scalar multiplication : Let $\lambda \in \mathbb{R}$ and $P(X) \in E_8$. Then $\deg(\lambda P) = \deg(P) \leq 2$:

$$\lambda P \in E_8$$

Thus, E_8 is closed under scalar multiplication.

Since E_8 is closed under vector addition and scalar multiplication, E_8 is a vector subspace.

Exercise 3 : $\mathbf{X}$ is a linear combination of $\mathbf{a}$ and $\mathbf{b}$ if and only if there exist α and $\beta \in \mathbb{R}$ such that $\mathbf{X} = \alpha\mathbf{a} + \beta\mathbf{b}$.

Then,

$$\mathbf{X} = \alpha\mathbf{a} + \beta\mathbf{b} \iff (1, -2, k) = \alpha(1, 1, 1) + \beta(1, 2, 3)$$

$$\iff \begin{cases} \alpha + \beta = 1 \\ \alpha + 2\beta = -2 \\ \alpha + 3\beta = k \end{cases}$$

$$\iff \begin{cases} \alpha = 4 \\ \beta = -3 \\ k = -5 \end{cases}$$

Exercise 4 :

1. We will show that $\{a, b\}, \{b, c\}$, and $\{a, c\}$ are linearly independent.

- $\{a, b\}$ is linearly independent $\iff \{\alpha a + \beta b = 0_{\mathbb{R}^3} \implies \alpha = \beta = 0\}$

$$\alpha a + \beta b = 0_{\mathbb{R}^3} \iff \alpha(0, 1, -1) + \beta(-1, 0, 1) = (0, 0, 0)$$

$$\iff \begin{cases} -\beta = 0 \\ \alpha = 0 \end{cases} \implies \alpha = 0, \beta = 0$$

- $\{b, c\}$ is linearly independent $\iff \{\alpha b + \beta c = 0_{\mathbb{R}^3} \implies \alpha = \beta = 0\}$

$$\alpha b + \beta c = 0_{\mathbb{R}^3} \iff \alpha(-1, 0, 1) + \beta(1, -1, 0) = (0, 0, 0)$$

$$\iff \begin{cases} -\alpha + \beta = 0 \\ -\beta = 0 \end{cases} \implies \alpha = 0, \beta = 0$$

- $\{a, c\}$ is linearly independent $\iff \{\alpha a + \beta c = 0_{\mathbb{R}^3} \implies \alpha = \beta = 0\}$

$$\alpha a + \beta c = 0_{\mathbb{R}^3} \iff \alpha(0, 1, -1) + \beta(1, -1, 0) = (0, 0, 0)$$

$$\iff \begin{cases} \beta = 0 \end{cases}$$

2. $\{a,b,c\}$ is free $\iff \{\alpha a+\beta b+\gamma c=0_{\mathbb{R}^3} \implies \alpha=\beta=\gamma=0\}$

$$\alpha a+\beta b+\gamma c=0_{\mathbb{R}^3} \iff \alpha(0,1,-1)+\beta(-1,0,1)+\gamma(1,-1,0)=(0,0,0)$$

$$\iff \begin{cases} -\beta+\gamma=0 \\ \alpha-\gamma=0 \\ -\alpha+\beta=0 \end{cases}$$

$$\iff \begin{cases} \gamma=\beta \\ \alpha=\gamma \\ \beta=\alpha \end{cases} \implies \alpha=\beta=\gamma \text{ but not necessarily zero}$$

Therefore, $\{a,b,c\}$ is not free.

3. Since the largest free set of $\{a,b,c\}$ contains two vectors, the subspace generated by $\{a,b,c\}$ has dimension equal to two.

Exercise 5 :

1. Let's check if the family $(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3)$ is linearly independent.

$$\lambda_1\mathbf{u}_1+\lambda_2\mathbf{u}_2+\lambda_3\mathbf{u}_3=\mathbf{0}\in\mathbb{R}^3 \iff \lambda_1(1,-1,2)+\lambda_2(1,1,1)+\lambda_3(-1,-5,1)=(0,0,0)$$

$$\begin{cases} \lambda_1+\lambda_2-\lambda_3=0 \\ -\lambda_1+\lambda_2-5\lambda_3=0 \\ 2\lambda_1+\lambda_2+\lambda_3=0 \end{cases}$$

Do this : $L_2=L_2+L_1, L_3=L_3-2L_1$

$$\begin{cases} \lambda_1+\lambda_2-\lambda_3=0 \\ 2\lambda_2-6\lambda_3=0 \\ -\lambda_2+3\lambda_3=0 \end{cases}$$

Combine L_2 and L_3, then :

$$\begin{cases} \lambda_1+\lambda_2-\lambda_3=0 \\ \lambda_2=3\lambda_3 \end{cases}$$

Substitute in L_1 :

$$\begin{cases} \lambda_1+3\lambda_3-\lambda_3=0 \\ \lambda_2=3\lambda_3 \end{cases}$$

We get :

$$\begin{cases} \lambda_1 = -2\lambda_3 \\ \lambda_2 = 3\lambda_3 \end{cases}$$

Therefore, the family is not free (is not linearly independent). Moreover, by taking $\lambda_3 = 1$, $\lambda_1 = -2$, and $\lambda_2 = 3$, we get :

$$-2\mathbf{u}_1 + 3\mathbf{u}_2 + \mathbf{u}_3 = \mathbf{0} \in \mathbb{R}^3$$

In other words :

$$\mathbf{u}_3 = 2\mathbf{u}_1 - 3\mathbf{u}_2$$

$$E = \text{span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = \text{span}(\mathbf{u}_1, \mathbf{u}_2, 2\mathbf{u}_1 - 3\mathbf{u}_2) = \text{span}(\mathbf{u}_1, \mathbf{u}_2)$$

Since the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are not proportional, they form a linearly independent and generating family for E, so they are a basis of E.

2. Since $0 + 0 + 0 = 0$, we have $\mathbf{0} \in F$.

   Let $\mathbf{u} = (x, y, z) \in F$ and $\mathbf{u}' = (x', y', z') \in F$, thus $x + y + z = 0$ and $x' + y' + z' = 0$.

   Let λ and λ' be any real numbers :

$$\lambda\mathbf{u} + \lambda'\mathbf{u}' = (\lambda x + \lambda' x', \lambda y + \lambda' y', \lambda z + \lambda' z') = (X, Y, Z)$$

$$X+Y+Z = (\lambda x+\lambda' x')+(\lambda y+\lambda' y')+(\lambda z+\lambda' z') = \lambda(x+y+z)+\lambda'(x'+y'+z') = \lambda\cdot 0+\lambda'\cdot 0 = 0$$

   This shows that $\lambda\mathbf{u} + \lambda'\mathbf{u}' \in F$.

   Therefore, F is a vector subspace of $\mathbb{R}^3$.

3. $\mathbf{u} = (x, y, z) \in F \iff x + y + z = 0 \iff z = -x - y$.

   Thus, $\mathbf{u} = (x, y, -x-y) = x(1, 0, -1) + y(0, 1, -1)$, with $\mathbf{a} = (1, 0, -1)$ and $\mathbf{b} = (0, 1, -1)$.

   $\{\mathbf{a}, \mathbf{b}\}$ is a generating set for F and since these two vectors are not proportional, they form a linearly independent family and thus, a basis of F.

4. Let $\mathbf{u} \in E \cap F$, there exist $\alpha, \beta, \gamma, \delta$ such that :

$$\mathbf{u} = \alpha\mathbf{u}_1 + \beta\mathbf{u}_2 = \gamma\mathbf{a} + \delta\mathbf{b}$$

$$\begin{cases} \alpha\mathbf{u}_1 + \beta\mathbf{u}_2 = \gamma\mathbf{a} + \delta\mathbf{b} \end{cases}$$

   Equating the coordinates :

$$\alpha(1, -1, 2) + \beta(1, 1, 1) = \gamma(1, 0, -1) + \delta(0, 1, -1)$$

   This gives the system of equations :

$$\begin{cases} \alpha + \beta - \gamma = 0 \\ -\alpha + \beta - \delta = 0 \\ 2\alpha + \beta + \gamma + \delta = 0 \end{cases}$$

First, we add L_2 to L_1 :

$$\begin{cases} \alpha + \beta - \gamma = 0 \\ (-\alpha + \beta - \delta) + (\alpha + \beta - \gamma) = 0 \\ 2\alpha + \beta + \gamma + \delta = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha + \beta - \gamma = 0 \\ 2\beta - \gamma - \delta = 0 \\ 2\alpha + \beta + \gamma + \delta = 0 \end{cases}$$

Now, subtract $2L_1$ from L_3 :

$$\begin{cases} \alpha + \beta - \gamma = 0 \\ 2\beta - \gamma - \delta = 0 \\ (2\alpha + \beta + \gamma + \delta) - 2(\alpha + \beta - \gamma) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha + \beta - \gamma = 0 \\ 2\beta - \gamma - \delta = 0 \\ -\beta + 3\gamma + \delta = 0 \end{cases}$$

Next, add $2L_3$ to L_2 :

$$\begin{cases} \alpha + \beta - \gamma = 0 \\ 2\beta - \gamma - \delta = 0 \\ 2(-\beta + 3\gamma + \delta) + (2\beta - \gamma - \delta) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha + \beta - \gamma = 0 \\ 2\beta - \gamma - \delta = 0 \\ 5\gamma + \delta = 0 \end{cases}$$

We now have a new system :

$$\Leftrightarrow \begin{cases} \alpha + \beta - \gamma = 0 \\ 2\beta - \gamma - \delta = 0 \\ \delta = -5\gamma \end{cases}$$

Solving this system :

$$\Leftrightarrow \begin{cases} \alpha + \beta - \gamma = 0 \\ \beta = -2\gamma \\ \delta = -5\gamma \end{cases}$$

Finally, we get :

$$\Leftrightarrow \begin{cases} \alpha = 3\gamma \\ \beta = -2\gamma \\ \delta = -5\gamma \end{cases}$$

Thus :

$$\mathbf{u} = \alpha\mathbf{u}_1 + \beta\mathbf{u}_2 = 3\gamma(1,-1,2) - 2\gamma(1,1,1) = \gamma(1,-5,4)$$

Also,

$$\mathbf{u} = \gamma\mathbf{a} + \delta\mathbf{b} = \gamma(1,0,-1) - 5\gamma(0,1,-1) = \gamma(1,-5,4)$$

.

Therefore, the intersection $E \cap F$ is spanned by the vector $\mathbf{c} = (1,-5,4)$:

$$E \cap F = \mathrm{span}(\mathbf{c})$$

Exercise 6 :

1.

$$\mathbf{u} = (x,y,z) \in E \iff \begin{cases} x + y - 2z = 0 \\ 2x - y - z = 0 \end{cases}$$

$$L_2 - 2L_1 : \begin{cases} x + y - 2z = 0 \\ -3y + 3z = 0 \end{cases}$$

$$\iff \begin{cases} x = z \\ y = z \end{cases}$$

$$\iff \mathbf{u} = (z,z,z) = z(1,1,1)$$

$$\iff \mathbf{u} = \lambda(1,1,1) \text{ for some } \lambda \in \mathbb{R}$$

Therefore, $E = \mathrm{span}(\mathbf{a})$, showing that E is a vector subspace of $\mathbb{R}^3$.

Alternative method :

$$\begin{cases} (0) + (0) - 2 \times (0) = 0 \\ 2 \times (0) - (0) - (0) = 0 \end{cases} \implies$$

$(0,0,0) \in E$

and

$$\begin{cases} x_2 + y_2 - 2z_2 = 0 \\ 2x_2 - y_2 - z_2 = 0 \end{cases}$$

Let $\mathbf{u}_1 = (x_1, y_1, z_1) \in E$ and $\mathbf{u}_2 = (x_2, y_2, z_2) \in E$, then

$$\lambda_1\mathbf{u}_1 + \lambda_2\mathbf{u}_2 = (\lambda_1 x_1 + \lambda_2 x_2, \lambda_1 y_1 + \lambda_2 y_2, \lambda_1 z_1 + \lambda_2 z_2)$$

$$(\lambda_1 x_1 + \lambda_2 x_2) + (\lambda_1 y_1 + \lambda_2 y_2) - 2(\lambda_1 z_1 + \lambda_2 z_2) = \lambda_1(x_1 + y_1 - 2z_1) + \lambda_2(x_2 + y_2 - 2z_2) = 0$$

$$2(\lambda_1 x_1 + \lambda_2 x_2) - (\lambda_1 y_1 + \lambda_2 y_2) - (\lambda_1 z_1 + \lambda_2 z_2) = \lambda_1(2x_1 - y_1 - z_1) + \lambda_2(2x_2 - y_2 - z_2) = 0$$

This shows that $\lambda_1\mathbf{u}_1 + \lambda_2\mathbf{u}_2 \in E$. Therefore, E is a vector subspace of $\mathbb{R}^3$.

2. $\{\mathbf{a}\}$ is a generating family of E. Since this vector is non-zero, it forms a basis for E, which is the line generated by the vector $\mathbf{a}$.

3. $$\mathbf{b} = (1,0,1), \quad \mathbf{c} = (0,1,1)$$

   These vectors are linearly independent and form a basis for F, thus $\dim(F) \geq 2$. Since $e_1 = (1,0,0) \notin F$, $F \subsetneq \mathbb{R}^3$, and hence $\dim(F) = 2$.

4. $$1 + 1 - 1 = 1 \neq 0 \implies \mathbf{a} \notin \text{span}(\mathbf{b}, \mathbf{c})$$

   The set $\{\mathbf{b}, \mathbf{c}\}$ is linearly independent, and therefore $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is also linearly independent. Since this set has three elements, it forms a basis for $\mathbb{R}^3$.

5. Since, $\text{span}(\mathbf{a}) \cap \text{span}(\mathbf{b}, \mathbf{c}) = 0_{\mathbb{R}^3}$. Other part, $\{\mathbf{a}\}$ is a basis for E, $\{\mathbf{b}, \mathbf{c}\}$ is a basis for F, and $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is a basis for $\mathbb{R}^3$. Then, $\forall u \in \mathbb{R}^3$, $\exists\ \alpha$, β and $\gamma \in \mathbb{R}$ $\mathbf{u} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$. Thus, $E + F = \mathbb{R}^3$. Therefore, $E \oplus F = \mathbb{R}^3$.

6. To find α, β, γ such that $\mathbf{u} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$:

$$\mathbf{u} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} \iff (x,y,z) = \alpha(1,1,1) + \beta(1,0,1) + \gamma(0,1,1)$$

$$\begin{cases} \alpha + \beta = x \\ \alpha + \gamma = y \\ \alpha + \beta + \gamma = z \end{cases}$$

$L_2 = L_2 - L_1, L_3 = L_3 - L_1$

$$\iff \begin{cases} \alpha + \beta = x \\ -\beta + \gamma = -x + y \\ \gamma = -x + z \end{cases}$$

$$\iff \begin{cases} \alpha = x - \beta \\ \beta = \gamma + x - y \\ \gamma = -x + z \end{cases}$$

$$\iff \begin{cases} \alpha = x + y - z \\ \beta = -y + z \\ \gamma = -x + z \end{cases}$$

Therefore,

$$\mathbf{u} = (x + y - z)\mathbf{a} + (-y + z)\mathbf{b} + (-x + z)\mathbf{c}$$