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Algebra 2 : Exercise Series N°01

Note : The (*) part is left for students.

Exercise 01 :

A- Let $E = \mathbb{R}^* \times \mathbb{R} = \{(a, b) : a, b \in \mathbb{R}, a \neq 0\}$ and $+$ be defined on E by

$$(a, b) + (c, d) = (ac, b + d)$$

1. Is $+$ a **vector addition** (Internal composition law) on E ?
2. Is $(E, +)$ a commutative group ?

B- Let an operation . be defined on E by :

$$\forall (a, b) \in E, \forall \lambda \in \mathbb{R} : \lambda.(a, b) = (a^\lambda, \lambda b)$$

1. Is (.) a **scalar multiplication** (External composition law) on E ?
2. Show that $(E, +, .)$ is a vector space over $\mathbb{R}$ ($\mathbb{R}$-vector space).

Exercise 02 : In $\mathbb{R}^3$ and $\mathbb{R}^4$, consider the following sets :

- $E_1 = \{(x, y) \in \mathbb{R}^2, x + y = 0\}$.
- $E_2 = \{(x, y, z) \in \mathbb{R}^3, x + y = z\}$.
- $E_3 = \{(x, y, z) \in \mathbb{R}^3, x - y + 5z = a,\ a \in \mathbb{R}\}$.
- $E_4 = \{(x, y, z) \in \mathbb{R}^3, z = 1\}$ (*).
- $E_5 = \{(x, y, z, t) \in \mathbb{R}^4, x + y + z + 2t = 0 \text{ and } x + y = 0\}$ (*).
- $E_6 = \{(x, y) \in \mathbb{R}^2, xy = 0\}$.
- $E_7 = \{(x, y) \in \mathbb{R}^2, y = x^2\}$ (*).
- $E_8 = \{P \in \mathbb{R}[X], \deg(P) \leq 2\}$.
- Are the sets $E_1, E_2, ..., E_8$ vector subspaces of $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^4$?

Exercise 03 :

- For which value(s) of k is the vector $\mathbf{X} = (1, -2, k)$ in $\mathbb{R}^3$ a linear combination of the vectors $\mathbf{a} = (1, 1, 1)$ and $\mathbf{b} = (1, 2, 3)$?

Exercise 04 :

- Let $\mathbf{a} = (0, 1, -1)$, $\mathbf{b} = (-1, 0, 1)$, and $\mathbf{c} = (1, -1, 0)$ be vectors in $\mathbb{R}^3$.

1. Show that these vectors are linearly independent in pairs.

2. Is the set $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ linearly independent ?

3. What is the dimension of the vector subspace spanned by the set $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$?

Exercise 05 : Let $\mathbf{u}_1 = (1, -1, 2)$, $\mathbf{u}_2 = (1, 1, 1)$, and $\mathbf{u}_3 = (-1, -5, 1)$. Let $E = \text{span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$. Let $F = \{(x, y, z) \in \mathbb{R}^3, x + y + z = 0\}$.

1. Find a basis for E.

2. Show that F is a vector subspace of $\mathbb{R}^3$.

3. Find a basis for F.

4. Find a basis for $E \cap F$.

Exercise 06 : Let $E = \{(x, y, z) \in \mathbb{R}^3, x + y - 2z = 0 \text{ and } 2x - y - z = 0\}$ and $F = \{(x, y, z) \in \mathbb{R}^3, x + y - z = 0\}$, two subsets of $\mathbb{R}^3$. We admit that F is a vector subspace of $\mathbb{R}^3$. Let $\mathbf{a} = (1, 1, 1)$, $\mathbf{b} = (1, 0, 1)$, and $\mathbf{c} = (0, 1, 1)$.

1. Show that E is a vector subspace of $\mathbb{R}^3$ (*).

2. Determine a generating set for E and show that this set is a basis.

3. Show that $\{\mathbf{b}, \mathbf{c}\}$ is a basis for F.

4. Show that $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is a linearly independent set in $\mathbb{R}^3$.

5. Determine whether $E \oplus F = \mathbb{R}^3$.

6. Let $\mathbf{u} = (x, y, z)$. Express $\mathbf{u}$ in the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$.

Exercise 07 : (Home-Work)

- Consider the family of vectors : $\mathbf{v_1} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}$, $\mathbf{v_2} = \begin{pmatrix} 3 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, which forms a basis of the vector subspace H of $\mathbb{R}^4$.

- Determine H.

Exercise 08 : (Home-Work)

Let $E = \{(x, y, z, t) \in \mathbb{R}^4, x + y + z + t = 0,\ x + 2y - z + t = 0,\ -x - y + 2z + 2t = 0\}$ and $F = \{(x, y, z, t) \in \mathbb{R}^4, x + 3y + 4t = 0\}$.

1. Find a basis for each of these two vector subspaces of $\mathbb{R}^4$.

2. Is $E \oplus F = \mathbb{R}^4$?

3. Let $a = (1, 3, 0, 4) \in \mathbb{R}^4$ and define $G = \text{Span}(a)$. Is $G \oplus F = \mathbb{R}^4$?