Exam of Algebra 2 (Normal Session) — **Duration : 02h**

Note : The use of calculators is strictly prohibited.

Exercise 01 : (8 pts)

I. Consider in $M_3(\mathbb{R})$ the following matrices :

$$A = \begin{pmatrix} 2 & 1 & 0 \\ -3 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

1. Compute : $B = I_3 - A$.
2. Show that $BA^2 + BA + B - I_3 = 0_3$ (zero matrix).
3. Deduce whether the matrix B is invertible, and determine its inverse B^{-1} if it exists.

II. Consider the following linear system S, where $m \in \mathbb{R}$:

$$(S) : \begin{cases} 4x + my + z = 4 \\ (-3-m)x - 2y - z = -5 \\ mx + y + z = 3 \end{cases}$$

1. Write the system S in matrix form $A_m X = b$.
2. For what values of m does the system S have a unique solution ?
3. Solve S for $m = 2$ using two methods :
   (a) The matrix inverse method.
   (b) Cramer's method.

Exercise 02 : (9 pts)

- Let f be the mapping defined from $\mathbb{R}^3$ into $\mathbb{R}^3$ by :

$$f(x, y, z) = (-x - y, 3x + 2y - z, -x + 2z),$$

and let $E = \{(x, y, z) \in \mathbb{R}^3, 2x - y = -5z\}$.

1. Show that f is linear.
2. Determine $\ker(f)$ and $\mathrm{Im}(f)$, and find their dimensions. Deduce that f is an isomorphism.
3. Prove that E is a vector subspace of $\mathbb{R}^3$. Determine a basis for E and compute its dimension.
4. Is $\ker(f) \oplus E = \mathbb{R}^3$?
5. Determine the formula for defining f^{-1}.

Exercise 03 : (3 pts)

Let E be a vector space over $\mathbb{R}$, and let $B = \{e_1, e_2, \ldots, e_n\}$ be a basis of E. Consider a linear map $f : E \longrightarrow E$.

- Prove that : $\mathrm{rank}(f) = n \Longrightarrow f$ is an automorphism.