Sample solution for Algebra 2 (Normal session)

Exercice 01 9 pts

I. 1

$$A = \begin{pmatrix} 2 & 1 & 0 \\ -3 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}; \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1. $B = I_3 - A = \begin{pmatrix} -1 & -1 & 0 \\ 3 & 2 & -1 \\ -1 & 0 & 2 \end{pmatrix}$ 0,5pt

2. $A^2 = A.A = \begin{pmatrix} 1 & 1 & 1 \\ -2 & -2 & -2 \\ 1 & 1 & 1 \end{pmatrix}$

So, $BA^2 = \begin{pmatrix} 1 & 1 & 1 \\ -2 & -2 & -2 \\ 1 & 1 & 1 \end{pmatrix}$

$BA = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 3 \\ 0 & -1 & -2 \end{pmatrix}$ 1pt

Then, $BA^2 + BA + B - I_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

3. Since $BA^2 + BA + B - I_3 = 0_3 \Rightarrow BA^2 + BA + B = I_3$
$\Rightarrow B(A^2 + A + I_3) = I_3$

By definition of the inverse matrix B is invertible (0,5 pt)

$$B^{-1} = A^2 + A + I_3 = \begin{pmatrix} 4 & 2 & 1 \\ -5 & -2 & -1 \\ 2 & 1 & 1 \end{pmatrix}$$ (1pt)

II. 1 $(S) \Leftrightarrow A_m X = b$

$$A_m = \begin{pmatrix} 4 & m & 1 \\ -3-m & -2 & -1 \\ m & 1 & 1 \end{pmatrix} ; \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} ; \quad b = \begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix}$$ (1pt)

2\. (S) has a unique solution $\Leftrightarrow A_m$ is invertible

$\Leftrightarrow \det(A_m) \neq 0$

- We compute $\det(A_m)$:

$$\det(A_m) = 4 \begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix} - m \begin{vmatrix} -3-m & -1 \\ m & 1 \end{vmatrix} + \begin{vmatrix} -3-m & -2 \\ m & 1 \end{vmatrix}$$ (1pt)

$$= -4 - m\big((-3-m) + m\big) + \big((-3-m) + 2m\big)$$

$$= -4 + 3m - 3 + m$$

$$= 4m - 7$$

$\Rightarrow \det(A_m) \neq 0 \Leftrightarrow m \neq \frac{1}{4}$.

Hence, (S) has a unique solution $\Leftrightarrow m \in \mathbb{R} \setminus \{1/4\}$

3\. For $m = 2$; $A_2 = \begin{pmatrix} 4 & 2 & 1 \\ -5 & -2 & -1 \\ 2 & 1 & 1 \end{pmatrix} = B^{-1}$ (qst 3) (0,5 pt)

- From question 3, $(B^{-1})^{-1} = B$. $A_2 X = b \Leftrightarrow B^{-1} X = b$

$\Leftrightarrow X = Bb$ (0,5 pt)

$$X = \begin{pmatrix} -1 & -1 & 0 \\ 3 & 2 & -1 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$ (1 pt)

(b) Cramer's method

Since the number of unknowns = the number of equations and $\boxed{\det(A_2) = 4(2) - 7 = 1} \neq 0 \Rightarrow$ we can apply Cramer's method. 0,5 pt

So, $x = \dfrac{\det(A_x)}{\det(A_2)}$; $A_x = \begin{pmatrix} 4 & 2 & 1 \\ -5 & -2 & -1 \\ 3 & 1 & 1 \end{pmatrix}$ 0,25 pt

$\det(A_x) = 4 \begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix} - 2 \begin{vmatrix} -5 & -1 \\ 3 & 1 \end{vmatrix} + 1 \begin{vmatrix} -5 & -2 \\ 3 & 1 \end{vmatrix}$

$= 1$ 0,25 pt

Then, $\boxed{x = 1}$

$y = \dfrac{\det(A_y)}{\det(A_2)}$; $A_y = \begin{pmatrix} 4 & 4 & 1 \\ -5 & -5 & -1 \\ 2 & 3 & 1 \end{pmatrix}$ 0,25 pt

$\det(A_y) = 4 \begin{vmatrix} -5 & -1 \\ 3 & 1 \end{vmatrix} - 4 \begin{vmatrix} -5 & -1 \\ 2 & 1 \end{vmatrix} + 1 \begin{vmatrix} -5 & -5 \\ 2 & 3 \end{vmatrix}$

$= -1$ 0,25 pt

Then, $\boxed{y = -1}$

$z = \dfrac{\det(A_z)}{\det(A_2)}$; $A_z = \begin{pmatrix} 4 & 2 & 4 \\ -5 & -2 & -5 \\ 2 & 1 & 3 \end{pmatrix}$ 0,25 pt

$\det(A_z) = 4 \begin{vmatrix} -2 & -5 \\ 1 & 3 \end{vmatrix} - 2 \begin{vmatrix} -5 & -5 \\ 2 & 3 \end{vmatrix} + 4 \begin{vmatrix} -5 & -2 \\ 2 & 1 \end{vmatrix}$

$= 2$ 0,25 pt

Then, $\boxed{z = 2}$

Exercise 2. 9 pts

1. f is linear: $\forall u=(x_1,x_2,x_3),\ v=(y_1,y_2,y_3)\in\mathbb{R}^3$

i) $f(u+v)=f((x_1,x_2,x_3)+(y_1,y_2,y_3))$

0,5 pt

$$= f(x_1+y_1,\ x_2+y_2,\ x_3+y_3)$$
$$= (-x_1-y_1-x_2-y_2,\ 3x_1+3y_1+2x_2+2y_2-x_3-y_3,\ -x_1-y_1+2x_3+2y_3)$$
$$= ((-x_1-x_2)+(-y_1-y_2),\ (3x_1+2x_2-x_3)+(3y_1+2y_2-y_3),\ (-x_1+2x_3)+(-y_1+2y_3))$$
$$= (-x_1-x_2,\ 3x_1+2x_2-x_3,\ -x_1+2x_3)+(-y_1-y_2,\ 3y_1+2y_2-y_3,\ -y_1+2y_3)$$
$$= f(u)+f(v).$$

ii) $\forall u=(x_1,x_2,x_3)\in\mathbb{R}^3,\ \forall\alpha\in\mathbb{R},$

0,5 pt

$$f(\alpha u)=f(\alpha x_1,\alpha x_2,\alpha x_3)=(-\alpha x_1-\alpha x_2,\ 3\alpha x_1+2\alpha x_2-\alpha x_3,\ -\alpha x_1+2\alpha x_3)$$
$$=\alpha(-x_1-x_2,\ 3x_1+2x_2-x_3,\ -x_1+2x_3)$$
$$=\alpha f(u)$$

2. i) $\mathrm{Ker}(f)=\{u(x,y,z)\ f(x,y,z)=0_{\mathbb{R}^3}\}$

$$f(x,y,z)=0_{\mathbb{R}^3}\iff\begin{cases}-x-y=0\\3x+2y-z=0\\-x+2z=0\end{cases}\iff\begin{cases}y=-x\\3x+2y-z=0\\x=2z\end{cases}$$
$$\iff \boxed{x=0;\ y=0;\ z=0}$$

1 pt

Then, $\mathrm{Ker}(f)=\{(0,0,0)\}$

Thus, $\boxed{\dim \operatorname{Ker}(f)=0}$ (0,5 pt)

ii) $\operatorname{Im}(f) = \{ f(x,y,z) \in \mathbb{R}^3 , (x,y,z) \in \mathbb{R}^3 \}$

$$= \{ (-x-y, 3x+2y-z, -x+2z) \in \mathbb{R}^3, (x,y,z) \in \mathbb{R}^3 \}$$

(1 pt)

$$= \left\{ x\begin{pmatrix}-1\\3\\-1\end{pmatrix} + y\begin{pmatrix}-1\\2\\0\end{pmatrix} + z\begin{pmatrix}0\\-1\\2\end{pmatrix} \in \mathbb{R}^3, (x,y,z) \in \mathbb{R}^3 \right\}$$

$$= \operatorname{span}\left(\begin{pmatrix}-1\\3\\-1\end{pmatrix}, \begin{pmatrix}-1\\2\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\2\end{pmatrix} \right)$$

By Rank theorem, $\dim \mathbb{R}^3 = \dim \operatorname{Ker}(f) + \dim \operatorname{Im}(f)$

$\Rightarrow \boxed{\dim \operatorname{Im}(f) = 3-0 = 3}$ (0,5 pt)

(0,25 pt)

iii) Since f is linear, injective (because $\operatorname{Ker}(f) = \{0_{\mathbb{R}^3}\}$) and surjective ($\dim \operatorname{Im}(f) = \dim \mathbb{R}^3 = 3$) (0,25)

Hence, f is linear and bijective $\Rightarrow$ f is an isomorphism

3_ E is a vector subspace of $\mathbb{R}^3$:

(1 pt)

(i) $(0,0,0) \in E$ $\left(2(0) - 0 = -5(0) \right)$

(ii) Let $u = (x_1, x_2, x_3) \in E$, $v = (y_1, y_2, y_3) \in E$

$u + v = (x_1+y_1, x_2+y_2, x_3+y_3)$

We have $2(x_1+y_1) - (x_2+y_2) = (2x_1 - x_2) + (2y_1 - y_2)$

$$= -5x_3 - 5y_3$$

$$= -5(x_3+y_3)$$

$\Rightarrow u + v \in E$

(iii) Let $u = (x_1, x_2, x_3) \in E$; $\alpha \in \mathbb{R}$

$\alpha u = (\alpha x_1, \alpha x_2, \alpha x_3)$. We have:

$2\alpha x_1 - \alpha x_2 = \alpha(2x_1 - x_2) = \alpha(-5x_3) = -5(\alpha x_3)$

$\Rightarrow \alpha u \in E$.

- Basis of E:

Let $u = (x, y, z) \in E \Rightarrow 2x - y = -5z \Rightarrow y = 2x + 5z$

$$\Rightarrow u = \begin{pmatrix} x \\ 2x+5z \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix}$$

Then, E is generated by $B = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix} \right\}$

Since B is free family (It's clear)

Thus B is basis of E. 1pt

- $\boxed{\dim E = 2.}$ 0,5pt

4. $Ker(f) \oplus E = \mathbb{R}^3 \Leftrightarrow \begin{cases} Ker(f) \cap E = \{0_{\mathbb{R}^3}\} \\ \text{and} \\ Ker(f) + E = \mathbb{R}^3 \end{cases}$ 0,5 pt

- Since $Ker(f) = \{0_{\mathbb{R}^3}\} \subset E$ (because, $0_{\mathbb{R}^3} \in E$).

$\Rightarrow Ker(f) \cap E = \{0_{\mathbb{R}^3}\}$

And, $\dim(Ker(f) + E) = \dim Ker(f) + \dim E - \dim(Ker(f) \cap E)$

$= 0 + 2 - 0 = 2 \neq \dim \mathbb{R}^3$

Thus, $\boxed{Ker(f) + E \neq \mathbb{R}^3} \Rightarrow \boxed{Ker(f) \oplus E \neq \mathbb{R}^3.}$ 0,5 pt

5. Since f is bijective $\Rightarrow \exists f^{-1} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$.

$f(x,y,z) = (y_1, y_2, y_3) \Longleftrightarrow f^{-1}(y_1, y_2, y_3) = (x, y, z)$ 0,5

Then, $\begin{cases} y_1 = -x-y \\ y_2 = 3x+2y-z \\ y_3 = -x+2z \end{cases} \iff \begin{cases} x = 4y_1+2y_2+y_3 \\ y = -5y_1-2y_2-y_3 \\ z = 2y_1+y_2+y_3 \end{cases}$ 0,5 pt

So, $\boxed{f^{-1}(y_1,y_2,y_3) = (4y_1+2y_2+y_3, -5y_1-2y_2-y_3, 2y_1+y_2+y_3)}$

Exercise 3. (2pts) not 3pts

- We suppose that $\operatorname{rank}(f) = n$. 0,5 pt

Since, $\dim E = n$ (number of vectors of B)

$\Rightarrow \operatorname{rank}(f) = \dim \operatorname{Im}(f) = \dim E$ 0,5 pt

$\Rightarrow \operatorname{Im}(f) = E \Rightarrow \boxed{f \text{ is surjective}}$

And by rank theorem: $\dim E = \dim \ker(f) + \operatorname{rank}(f)$ 0,5 pt

$\Rightarrow \dim \ker(f) = n - n = 0$

$\Rightarrow \ker(f) = \{0_E\}$

$\Rightarrow \boxed{f \text{ is injective}}$

Thus $f: E \longrightarrow E$ is linear mapping and bijective (endomorphism + bijective) $\Rightarrow f$ is bijective automorphism 0,5 pt