Course of Algebra 2

Chapter 04 : Solving Of Linear Equations System

4.1 Definitions

The set $\mathbb{K}$ always refers to $\mathbb{R}$ or $\mathbb{C}$.

Definition 1. *Let $n, p \geq 1$ be integers. A linear system of n linear equations with p unknowns is a list of n linear equations of the form :*

$$(S)\begin{cases} a_{11}x_1 + \cdots + a_{1p}x_p = b_1 \\ a_{21}x_1 + \cdots + a_{2p}x_p = b_2 \\ \vdots \\ a_{n1}x_1 + \cdots + a_{np}x_p = b_n \end{cases}$$

*Where $(x_j)_{j=1,\ldots,p}$ are the unknowns, and $(a_{ij}), b_j \in \mathbb{K}$. a_{ij} are called the **coefficients** of the linear system*

Definition 2.
- *The **homogeneous system** associated with S is obtained by replacing each b_j with 0.*
- *A **solution** of S is a p-tuple $(x_1, x_2, \ldots, x_p)$ that simultaneously satisfies the n equations of S.*
- ***Solving** S means finding all possible solutions that satisfy these equations.*

Remark 4.1.1. *A linear system may not have a solution at all. If this is the case, we say that the linear system is **inconsistent** :*

$$\textit{INCONSISTENT} \iff \textit{NO SOLUTION}$$

*A linear system is called **consistent** if it has at least one solution :*

$$\textit{CONSISTENT} \iff \textit{AT LEAST ONE SOLUTION}$$

Example 4.1.2. *Show that the linear system does not have a solution :*

$$\begin{cases} -x_1 + x_2 = 3 \\ x_1 - x_2 = 1 \end{cases}$$

solution *Adding the two equations results in :*

$$0 = 4$$

which is a contradiction. Therefore, there does not exist a list (s_1, s_2) that satisfies the system, as this would lead to the contradiction $0 = 4$.

Example 4.1.3. *Consider the linear system :*

$$\begin{cases} x_1 + x_2 = 4 \\ 2x_1 + 2x_2 = 8 \end{cases}$$

solution *Dividing the second equation by 2 yields :*

$$x_1 + x_2 = 4$$

which is the same as the first equation. Since we have only one equation with two unknowns, there are infinitely many solutions parameterized by $x_1 = 4 - x_2$, where x_2 is a free variable.

Definition 3 (**Matrix Form**). *Let us define the vectors :*

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & \dots & a_{1p} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{np} \end{pmatrix}$$

The system (S) is equivalent to the matrix equation :

$$AX = B$$

If f is a linear map from $\mathbb{K}^p$ to $\mathbb{K}^n$ such that A is the matrix associated with f in canonical bases, then the system (S) becomes :

$$f(X) = B$$

Example 4.1.4. *Consider the following linear systems :*

$$(S_1) \begin{cases} 2x + y = 5 \\ x - 3y = -4 \\ 4x + 2y = 10 \end{cases}$$

Writing the system in matrix form :

$$\begin{pmatrix} 2 & 1 \\ 1 & -3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ 10 \end{pmatrix}$$

$$(S_2) \begin{cases} x + y + z = 2 \\ 2x - y + 3z = 5 \\ 4x + 2y + z = 7 \end{cases}$$

Writing the system in matrix form :

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix}$$

4.2 Methods for Solving a Linear System

4.2.1 Using Inverse Matrix A^{-1}

Definition 4. *A square linear system (S) in matrix form $AX = B$ admits the solution :*

$$X = A^{-1}B \quad \textit{if } A \textit{ is invertible.}$$

Theorem 4.2.1. *Let $AX = B$ be a system of linear equations, where A is an $n \times n$ matrix and X is a column vector of unknowns. The system has a unique solution if and only if A is invertible, which is equivalent to :*

$$\det(A) \neq 0.$$

In this case, the unique solution is given by :

$$X = A^{-1}B.$$

Example 4.2.2. *Consider the following linear system :*

$$\begin{cases} 10x + 5y + 4z = 1 \\ 5x + 2y + 3z = 2 \\ 3x + y + 2z = 3 \end{cases}$$

(1) Write the system S in matrix form $AX = B$.

(2) Prove that the system S has a unique solution.

(3) Solve the system using the inverse matrix method.

Solution

1. ***Writing the system in matrix form :*** *The given system can be expressed as :*

$$AX = B$$

where :

$$A = \begin{pmatrix} 10 & 5 & 4 \\ 5 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

2. ***Proving that the system has a unique solution :*** *A system of linear equations has a unique solution if and only if A is invertible, which requires* $\det(A) \neq 0$.

Computing the determinant :

$$\det(A) = \begin{vmatrix} 10 & 5 & 4 \\ 5 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix}$$

Expanding along the first row :

$$\det(A) = 10 \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} - 5 \begin{vmatrix} 5 & 3 \\ 3 & 2 \end{vmatrix} + 4 \begin{vmatrix} 5 & 2 \\ 3 & 1 \end{vmatrix}$$

Computing minors :

$$\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = (2 \times 2 - 3 \times 1) = 4 - 3 = 1$$

$$\begin{vmatrix} 5 & 3 \\ 3 & 2 \end{vmatrix} = (5 \times 2 - 3 \times 3) = 10 - 9 = 1$$

$$\begin{vmatrix} 5 & 2 \\ 3 & 1 \end{vmatrix} = (5 \times 1 - 2 \times 3) = 5 - 6 = -1$$

Thus :

$$\det(A) = (10 \times 1) - (5 \times 1) + (4 \times -1) = 10 - 5 - 4 = 1 \neq 0$$

Since $\det(A) \neq 0$*, the matrix A is invertible, ensuring the existence and uniqueness of the solution.*

3. ***Solving the system using the inverse matrix method :*** *Since A^{-1} is given as :*

$$A^{-1} = \frac{1}{det(A)}.Cof(A)^T = \begin{pmatrix} 1 & -6 & 7 \\ -1 & 8 & -10 \\ -1 & 5 & -5 \end{pmatrix}$$

We compute :

$$X = A^{-1}B$$

Performing matrix-vector multiplication :

$$X = \begin{pmatrix} 1 & -6 & 7 \\ -1 & 8 & -10 \\ -1 & 5 & -5 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Computing each component :

$$x = (1 \times 1) + (-6 \times 2) + (7 \times 3) = 1 - 12 + 21 = 10$$

$$y = (-1 \times 1) + (8 \times 2) + (-10 \times 3) = -1 + 16 - 30 = -15$$

$$z = (-1 \times 1) + (5 \times 2) + (-5 \times 3) = -1 + 10 - 15 = -6$$

Final solution :

$$x = 10, \quad y = -15, \quad z = -6.$$

4.2.2 Using Cramer's Method

Definition 5. *A system (S) is called a* ***Cramer system*** *if $n = p = r$, meaning that (S) is a system of n equations with n unknowns and satisfies :*

$$\det(A) \neq 0$$

where r is the rank of the matrix representing the system.

The solution of a **Cramer system** given in matrix form $AX = B$ is determined by :

$$x_j = \frac{\det(A_j)}{\det(A)}, \quad 1 \leq j \leq n,$$

where A_j is the matrix obtained from A by replacing the j-th column with the column of constants B.

Example 4.2.3. *Consider the following linear system :*

$$\begin{cases} 10x + 5y + 4z = 1 \\ 5x + 2y + 3z = 2 \\ 3x + y + 2z = 3 \end{cases}$$

Solution Using Cramer's Method to solve the previous example :

$$x_j = \frac{\det(A_j)}{\det(A)}, \quad 1 \leq j \leq n$$

Since $\det(A) = 1 \neq 0$, the system has a unique solution.

1. **Compute x :** Replace the first column of A with B :

$$A_1 = \begin{pmatrix} 1 & 5 & 4 \\ 2 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

Computing $\det(A_1)$:

$$\det(A_1) = (1 \times 1) - (5 \times -5) + (4 \times -4) = 10$$

$$x = \frac{\det(A_1)}{\det(A)} = \frac{10}{1} = 10$$

2. **Compute y :** Replace the second column of A with B :

$$A_2 = \begin{pmatrix} 10 & 1 & 4 \\ 5 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$$

Computing $\det(A_2)$:

$$\det(A_2) = -15$$

$$y = \frac{\det(A_2)}{\det(A)} = \frac{-15}{1} = -15$$

3. **Compute z :** Replace the third column of A with B :

$$A_3 = \begin{pmatrix} 10 & 5 & 1 \\ 5 & 2 & 2 \\ 3 & 1 & 3 \end{pmatrix}$$

Computing $\det(A_3)$:

$$\det(A_3) = -6$$

$$z = \frac{\det(A_3)}{\det(A)} = \frac{-6}{1} = -6$$

Final solution :

$$x = 10, \quad y = -15, \quad z = -6.$$