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Course of Algebra 2

Chapter 03 : Matrices

0.1 Definitions

Definition 1. *Let $\mathbb{K}$ be a field, denoting $\mathbb{R}$ or $\mathbb{C}$. A matrix in $\mathbb{K}$ of type (n,p) is called a rectangular array A of elements of $\mathbb{K}$ with n rows and p columns.*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{pmatrix}$$

Let a_{ij} denote the element located in the i-th row and the j-th column of a matrix A. The matrix A is denoted as

$$A = (a_{ij})_{1\leq i\leq n,\, 1\leq j\leq p}.$$

The set of all matrices of type (n,p) over the field $\mathbb{K}$ is denoted by $M(n,p)(\mathbb{K})$.

Special Types of Matrices :

1. *For $n = 1$, A is called a row matrix and is written as*

$$A = (a_{11}, a_{12}, \ldots, a_{1p}).$$

2. *For $p = 1$, A is called a column matrix and is written as*

$$A = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}.$$

3. *For $n = p$, A is called a square matrix of order n and is denoted as $A \in M_n(\mathbb{K})$.*

Example 1.2.

1. $A_1 = \begin{pmatrix} 1 & 4 & 0 \\ 2 & 3 & 0 \\ 3 & 2 & 0 \\ 4 & 1 & 3 \end{pmatrix}$ *is a matrix of type $(4,3)$.*

2. $A_2 = \begin{pmatrix} -5 & 2 & 1 & 0 \\ 1 & 0 & 1 & 7 \end{pmatrix}$ *is a matrix of type $(2,4)$.*

3. $A_3 = \begin{pmatrix} -1 & 9 \\ 6 & 0 \end{pmatrix}$ *is a square matrix of order 2.*

Definition 1.3. *Let $A = (a_{ij})_{1\leq i\leq n,\, 1\leq j\leq p}$ and $B = (b_{ij})_{1\leq i\leq n,\, 1\leq j\leq p}$ be two matrices of type (n,p).*

1. *$A = B$ if and only if $\forall i = 1, \ldots, n$ and $\forall j = 1, \ldots, p$, $a_{ij} = b_{ij}$.*

2. *The transpose of the matrix A, denoted by A^T, is the matrix defined as*

$$A^T = (a_{ji})_{1\leq j\leq p,\, 1\leq i\leq n},$$

which is the matrix of type (p, n) obtained by interchanging the rows and columns of A. Furthermore, $(A^T)^T = A$.

Example 1.4.

1. *For* $A_1 = \begin{pmatrix} 1 & 4 & 0 \\ 2 & 3 & 0 \\ 3 & 2 & 0 \\ 4 & 1 & 3 \end{pmatrix}$, *the transpose is* $A_1^T = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$.

2. *For* $A_2 = \begin{pmatrix} -1 & 2 & 1 & 8 \\ 1 & 0 & 0 & -5 \end{pmatrix}$, *the transpose is* $A_2^T = \begin{pmatrix} -1 & 1 \\ 2 & 0 \\ 1 & 0 \\ 8 & -5 \end{pmatrix}$.

3. *For* $A_3 = \begin{pmatrix} 1 & 0 \\ 5 & -5 \end{pmatrix}$, *the transpose is* $A_3^T = \begin{pmatrix} 1 & 5 \\ 0 & -5 \end{pmatrix}$.

1. Vector Space Structure on $M(n, p)(\mathbb{K})$

We define the following operations on $M(n, p)(\mathbb{K})$, where $\mathbb{K}$ is a field (typically $\mathbb{R}$ or $\mathbb{C}$).

Addition : Define

$$+ \;:\; M(n, p)(\mathbb{K}) \times M(n, p)(\mathbb{K}) \to M(n, p)(\mathbb{K})$$

by

$$\left(\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix} \right) \mapsto \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1p}+b_{1p} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2p}+b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & \cdots & a_{np}+b_{np} \end{pmatrix}.$$

Scalar Multiplication : Define

$$\cdot \;:\; \mathbb{K} \times M(n, p)(\mathbb{K}) \to M(n, p)(\mathbb{K})$$

by

$$\left(\lambda, \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{pmatrix} \right) \mapsto \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1p} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{n1} & \lambda a_{n2} & \cdots & \lambda a_{np} \end{pmatrix}.$$

Thus, $(M(n, p)(\mathbb{K}), +, \cdot)$ is a vector space over $\mathbb{K}$ of dimension $n \times p$. The additive identity is the zero matrix

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

2. Product of Two Matrices

Definition 2.1. Let

$$A \in M(n,p)(\mathbb{K}) \quad \text{and} \quad B \in M(p,m)(\mathbb{K}).$$

The product of the matrix A by B is defined as the matrix

$$C = (c_{ij})_{1 \le i \le n,\ 1 \le j \le m} \in M(n,m)(\mathbb{K}),$$

with

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj}.$$

Remark 2.2.

1. The element c_{ij} of the matrix C is calculated by summing the products of the entries of the i-th row of A with the corresponding entries of the j-th column of B.
2. The product $A \cdot B$ is defined only if the number of columns of A equals the number of rows of B.

Example 2.3. Consider

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \end{pmatrix} \in M(2,3)(\mathbb{K}) \quad \text{and} \quad B = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \in M(3,4)(\mathbb{K}).$$

Since A is of type $(2,3)$ and B is of type $(3,4)$, the product $C = A \cdot B$ is of type $(2,4)$ and is computed as :

$$C = A \cdot B = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \end{pmatrix},$$

where

$$\begin{aligned} c_{11} &= 1 \cdot 1 + 1 \cdot 2 + 0 \cdot 1 = 3, \\ c_{12} &= 1 \cdot 2 + 1 \cdot 0 + 0 \cdot 1 = 2, \\ c_{13} &= 1 \cdot 0 + 1 \cdot 1 + 0 \cdot 0 = 1, \\ c_{14} &= 1 \cdot 1 + 1 \cdot 1 + 0 \cdot 0 = 2, \\ c_{21} &= 2 \cdot 1 + 2 \cdot 2 + 0 \cdot 1 = 6, \\ c_{22} &= 2 \cdot 2 + 2 \cdot 0 + 0 \cdot 1 = 4, \\ c_{23} &= 2 \cdot 0 + 2 \cdot 1 + 0 \cdot 0 = 2, \\ c_{24} &= 2 \cdot 1 + 2 \cdot 1 + 0 \cdot 0 = 4. \end{aligned}$$

Thus,

$$C = \begin{pmatrix} 3 & 2 & 1 & 2 \\ 6 & 4 & 2 & 4 \end{pmatrix}.$$

Remark 2.4 : Non-Commutativity of Matrix Multiplication
The product of two matrices is not commutative. For example :

$$A \cdot B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix},$$

while

$$B \cdot A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}.$$

3. Square Matrices

Definition 3.1. Let A be a square matrix of order n, $A = (a_{ij})_{1\leq i\leq n,\, 1\leq j\leq n}$:

1. The sequence of elements $\{a_{11}, a_{22}, \ldots, a_{nn}\}$ is called the principal diagonal of A.
2. The trace of A is the number
$$\mathrm{Tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}.$$
3. A is called a diagonal matrix if $a_{ij} = 0$ for all $i \neq j$, i.e., all elements of A are zero except those on the principal diagonal.
4. A is called an upper triangular matrix (resp. lower triangular) if $a_{ij} = 0$ for all $i > j$ (resp. $i < j$), meaning the elements below (resp. above) the diagonal are zero.
5. A is called a symmetric matrix if $A = A^T$, where A^T is the transpose of A.

Example 3.2.

1. $A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is a diagonal matrix.
2. $A_2 = \begin{pmatrix} -5 & 0 & 0 \\ 1 & 0 & 0 \\ 6 & 3 & 9 \end{pmatrix}$ is a lower triangular matrix.
3. $A_3 = \begin{pmatrix} 7 & 40 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix}$ is an upper triangular matrix.
4. $A_4 = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 12 & 1 \\ -2 & 1 & 10 \end{pmatrix}$, where $A_4 = A_4^T$, is a symmetric matrix.

Proposition 3.3. Matrix multiplication is a closed operation in $M(n,n)(\mathbb{K})$ and has a neutral element called the identity matrix, denoted by I_n, defined as :

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

4. Determinants

Definition 4.1. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

be a matrix in $M(2,2)(\mathbb{K})$. The determinant of A is the real number given by :

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

It is denoted by $\det(A)$ or

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

Example 4.2. Calculate the determinant of A :

$$\det(A) = \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} = 1(-1) - 0(2) = -1.$$

Definition 4.3. Similarly, the determinant of a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in M_3(\mathbb{K})$$

is defined as :

$$\det(A) = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Example 4.4. Calculate the determinant of A :

$$\det(A) = \begin{vmatrix} 1 & 0 & -1 \\ 12 & -1 & 1 \\ 0 & 1 & 0 \end{vmatrix} = (-1)^{1+1} \cdot 1 \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} + (-1)^{1+2} \cdot 0 \begin{vmatrix} 12 & 1 \\ 0 & 0 \end{vmatrix} + (-1)^{1+3} \cdot (-1) \begin{vmatrix} 12 & -1 \\ 0 & 1 \end{vmatrix}.$$

Simplifying, we have :

$$\det(A) = -1 + 0 - 12 = -13.$$

Remark 4.5. To compute the determinant of a matrix A, one can expand A along any row or column. Based on this proposition, it is advantageous to choose the row or column with the most zeros.

Determinants of $n \times n$ Matrices

Using the 3×3 case as a guide, we define the determinant of a general $n \times n$ matrix as follows :

Definition 4.6 : Determinants by Row or Column Let $A = (a_{ij})_{1 \le i \le n,\, 1 \le j \le n}$. The determinant along the j-th column is given by :

$$\det(A) = (-1)^{1+j} a_{1j} D_{1j} + (-1)^{2+j} a_{2j} D_{2j} + \cdots + (-1)^{n+j} a_{nj} D_{nj}, \quad j = 1, \ldots, n.$$

The determinant along the i-th row is given by :

$$\det(A) = (-1)^{i+1} a_{i1} D_{i1} + (-1)^{i+2} a_{i2} D_{i2} + \cdots + (-1)^{i+n} a_{in} D_{in}, \quad i = 1, \ldots, n.$$

Here, A_{ij} represents the determinant minor of the term a_{ij}, which is the determinant of order $n-1$ obtained from $\det(A)$ by deleting the i-th row and the j-th column.

Properties of Determinants

Proposition 4.7. Let $A \in M_n(\mathbb{K})$. We have the following properties :

1. $\det(A) = \det(A^T)$, where A^T is the transpose of A.
2. $\det(A) = 0$ if two rows (or two columns) of A are identical.
3. $\det(A) = 0$ if two rows (or two columns) of A are proportional.
4. $\det(A) = 0$ if one row is a linear combination of two other rows of A (the same holds for columns).
5. The determinant of A does not change if a row is replaced by a linear combination of other rows (the same applies to columns).
6. If $B \in M_n(\mathbb{K})$, then $\det(A \cdot B) = \det(A) \cdot \det(B)$.

7. If A is a diagonal or triangular matrix, the determinant of A is the product of the elements on its principal diagonal $(a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn})$.

Remark 4.8. When n permutations of rows or columns are performed, the determinant is multiplied by $(-1)^n$.

Examples 4.9.

(1) Let

$$A = \begin{bmatrix} 3 & 0 & -5 \\ 2 & 2 & 1 \\ 3 & 0 & -5 \end{bmatrix}$$

Since row 1 is equal to row 3 ($L_1 = L_3$), we have :

$$|A| = 0.$$

(2) Let

$$B = \begin{bmatrix} 9 & 0 & -15 & 3 \\ 2 & 2 & 1 & 1 \\ 1 & 0 & -1 & 4 \\ 3 & 0 & -5 & 1 \end{bmatrix}$$

Since row 1 is equal to 3$\cdot$ row 4 ($L_1 = 3 \cdot L_4$), we have :

$$|B| = 0.$$

(3) Let

$$C = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 1 & 1 & 2 & 20 \\ 0 & 0 & -1 & 4 \\ 1 & 1 & -10 & 2 \end{bmatrix}$$

Since column 1 is equal to column 2 ($C_1 = C_2$), we have :

$$|C| = 0.$$

Example of Row Permutation and Determinant 4.10.
Consider the original matrix :

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \\ 5 & 2 & 6 \end{bmatrix}$$

The determinant of A is computed as :

$$\det(A) = 1 \cdot (1 \cdot 6 - 4 \cdot 2) - 0 \cdot (3 \cdot 6 - 4 \cdot 5) + 2 \cdot (3 \cdot 2 - 1 \cdot 5)$$

Simplifying :

$$\det(A) = 1 \cdot (-2) + 0 \cdot (\cdots) + 2 \cdot (6 - 5) = -2 + 2 = -4$$

Now, swap the first and third rows to form the new matrix A' :

$$A' = \begin{bmatrix} 5 & 2 & 6 \\ 3 & 1 & 4 \\ 1 & 0 & 2 \end{bmatrix}$$

Determinant After Row Swap
Swapping rows introduces one permutation, which multiplies the determinant by -1 :

$$\det(A') = -\det(A) = -(-4) = 4$$

The determinant of the new matrix A' is :

$$\det(A') = 4$$

Definition 4.11. Let $V_1, V_2, \ldots, V_n$, n vectors in $\mathbb{R}^n$. The determinant of the vectors $(V_1, V_2, \ldots, V_n)$, denoted $\det(V_1, V_2, \ldots, V_n)$, is the determinant of the matrix whose columns are the vectors $V_1, V_2, \ldots, V_n$.

Example 4.12. Let $V_1 = (1, 1, 0)$, $V_2 = (0, -1, 1)$, $V_3 = (0, 0, 1)$, then :

$$\det(V_1, V_2, V_3) = \begin{vmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = +1 - \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

Proposition 4.13. Let $V_1, V_2, \ldots, V_n$, n vectors in $\mathbb{R}^n$. The vectors $(V_1, V_2, \ldots, V_n)$ form a basis of $\mathbb{R}^n$ if and only if $\det(V_1, V_2, \ldots, V_n) \neq 0$.

Example 4.14 : Let $V_1 = (1, 2, 0)$, $V_2 = (0, -1, 1)$, $V_3 = (0, 0, 1)$. The vectors form a basis of $\mathbb{R}^3$, because :

$$\det(V_1, V_2, V_3) = \begin{vmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = -1 \neq 0$$

4.1. The Rank of a Matrix

Definition 4.15. Let $A \in M_{n,p}(\mathbb{K})$. The rank of A, denoted by $\mathrm{rg}(A)$, is the order of the largest square submatrix B extracted from A such that $\det(B) \neq 0$.

Example 4.16.

1. Let
$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \quad \det(A) = 2 \neq 0, \quad \mathrm{rg}(A) = 2.$$

2. Let
$$B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \det(B) = 0, \quad \mathrm{rg}(B) = 1.$$

3. Let
$$C = \begin{bmatrix} 0 & 1 & -1 & 0 \\ -1 & 1 & -1 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix},$$

   where $\mathrm{rg}(C) < 4$ and $\mathrm{rg}(C) \leq 3$. The largest square submatrix extracted from C is of order 3. In this example, there are four possibilities :

$$C_1 = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix},$$

$$C_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad C_4 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

   For these submatrices :

$$\det(C_1) = \det(C_2) = 0, \quad \det(C_3) = \det(C_4) = 0.$$

   Hence, $\mathrm{rg}(C) < 3$, and considering :

$$\begin{vmatrix} -1 & 0 \\ -1 & 1 \end{vmatrix} = -1 \neq 0,$$

   we conclude that $\mathrm{rg}(C) = 2$.

Theorem 4.17. The rank of a matrix is equal to the maximum number of linearly independent row (or column) vectors.

5. Invertible Matrices

Definition 5.1. A matrix $A \in M(n,n)(\mathbb{K})$ is said to be invertible if there exists a matrix $B \in M(n,n)(\mathbb{K})$ such that

$$A \cdot B = B \cdot A = I_n.$$

Example 5.2.
Let us show that the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$$

is invertible by finding a matrix $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$A \cdot B = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2,$$

or equivalently

$$B \cdot A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Solving the system yields :

$$B = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}.$$

Cofactor Matrix

Definition 5.3. Let $A = (a_{ij})_{1\le i\le n, 1\le j\le n} \in M_n(\mathbb{K})$. The cofactor of A at index i,j is the scalar :

$$c_{ij} = (-1)^{i+j}\det(A_{ij}),$$

where A_{ij} is the matrix obtained from A by removing row i and column j. The matrix $C = (c_{ij})_{1\le i\le n, 1\le j\le n}$ is called the cofactor matrix, and C^T (the transpose of C) is called the adjugate matrix of A.

Example 5.4. Let the matrix :

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & -1 & 1 \\ 0 & 2 & 2 \end{bmatrix}.$$

We calculate the cofactors of A :

$$c_{11} = (-1)^{1+1}\det(A_{11}) = (-1)^2 \begin{vmatrix} -1 & 1 \\ 2 & 2 \end{vmatrix} = -4.$$

$$c_{12} = (-1)^{1+2}\det(A_{12}) = (-1)^3 \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = -2.$$

$$c_{13} = (-1)^{1+3}\det(A_{13}) = (-1)^4 \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2.$$

Similarly :

$$c_{21} = (-1)^{2+1}\det(A_{21}) = 6, \quad c_{22} = (-1)^{2+2}\det(A_{22}) = 2, \quad c_{23} = (-1)^{2+3}\det(A_{23}) = -2.$$

$$c_{31} = (-1)^{3+1}\det(A_{31}) = 3, \quad c_{32} = (-1)^{3+2}\det(A_{32}) = 2, \quad c_{33} = (-1)^{3+3}\det(A_{33}) = -1.$$

Thus, the cofactor matrix is :

$$C = \begin{bmatrix} -4 & -2 & 2 \\ 6 & 2 & -2 \\ 3 & 2 & -1 \end{bmatrix},$$

and the adjugate matrix is :

$$C^T = \begin{bmatrix} -4 & 6 & 3 \\ -2 & 2 & 2 \\ 2 & -2 & -1 \end{bmatrix}.$$

Theorem 5.5. Let $A \in M_n(\mathbb{K})$. The matrix A is invertible if and only if $\det(A) \neq 0$. In this case, the inverse of A is given by :

$$A^{-1} = \frac{1}{\det(A)} C^T,$$

where C^T is the adjugate matrix of A.
Example 5.6. Let :

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & -1 & 1 \\ 0 & 2 & 2 \end{bmatrix}.$$

The determinant of A is :

$$\det(A) = 2 \neq 0,$$

so A is invertible. Furthermore :

$$A^{-1} = \frac{1}{2} C^T = \frac{1}{2} \begin{bmatrix} -4 & 6 & 3 \\ -2 & 2 & 2 \\ 2 & -2 & -1 \end{bmatrix},$$

which simplifies to :

$$A^{-1} = \begin{bmatrix} -2 & 3 & \frac{3}{2} \\ -1 & 1 & 1 \\ 1 & -1 & -\frac{1}{2} \end{bmatrix}.$$

We can verify that $A^{-1}A = I_3 = AA^{-1}$.

6. Relations Between a Linear Map and Its Associated Matrix

Definition 6.1 The matrix $A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ is called the matrix of f with respect to the bases B and B', and is sometimes denoted as $M_{(B,B')}(f)$. If $E = F$ and $B = B'$, then A is simply referred to as the matrix of f with respect to the base B and is denoted as $M_{(B)}(f)$.
Example 6.2

1. Consider the linear map :

$$f : \mathbb{R}^3 \to \mathbb{R}^2, \quad (x, y, z) \mapsto (x + y + z, x - y)$$

The canonical basis of $\mathbb{R}^3$ is given by :

$$B = \{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$$

The canonical basis of $\mathbb{R}^2$ is :

$$B' = \{v_1 = (1, 0), v_2 = (0, 1)\}$$

Applying f to the basis vectors :

$$f(e_1) = f(1, 0, 0) = (1, 1) = v_1 + v_2$$

$$f(e_2) = f(0, 1, 0) = (1, -1) = v_1 - v_2$$

$$f(e_3) = f(0, 0, 1) = (1, 0) = v_1$$

The matrix representation is :

$$M_{(B,B')}(f) = \begin{bmatrix} f(e_1) & f(e_2) & f(e_3) \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

2. Consider the linear map :

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y) \mapsto (x + y, x - y)$$

The bases are :

$$B = \{e_1 = (1, 2), e_2 = (-1, 1)\}, \quad B' = \{v_1 = (0, 2), v_2 = (-2, 1)\}$$

We need to determine the coefficients $\lambda_1, \lambda_2, \lambda_3, \lambda_4$:

$$f(e_1) = f(1, 2) = (3, -1) = \lambda_1 v_1 + \lambda_2 v_2$$

$$f(e_2) = f(-1, 1) = (0, -2) = \lambda_3 v_1 + \lambda_4 v_2$$

Solving for the coefficients :

$$\begin{cases} \lambda_1 = \frac{1}{4}, & \lambda_2 = -\frac{3}{2} \\ \lambda_3 = -1, & \lambda_4 = 0 \end{cases}$$

The matrix representation is :

$$M_{(B,B')}(f) = \begin{bmatrix} \lambda_1 & \lambda_3 \\ \lambda_2 & \lambda_4 \end{bmatrix} = \begin{bmatrix} f(e_1) & f(e_2) \\ \frac{1}{4} & -1 \\ -\frac{3}{2} & 0 \end{bmatrix}$$

Proposition 6.3 Let E and F be two finite-dimensional vector spaces over $\mathbb{K}$, with dimensions n and m. Let $B = (e_1, e_2, ..., e_n)$ be a basis of E, and $B' = (v_1, v_2, ..., v_m)$ a basis of F. Then, given a matrix $A \in M(n, m)(\mathbb{K})$, there exists a unique linear map $f : E \to F$ whose matrix with respect to the bases B and B' is A.

Example 6.4 Consider the matrix :

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$$

This represents the linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ with respect to the canonical basis (e_1, e_2). Applying f to the basis vectors :

$$f(e_1) = e_1 + 2e_2 \Rightarrow f(1, 0) = (1, 0) + 2(0, 1) = (1, 2)$$

$$f(e_2) = -e_1 \Rightarrow f(0, 1) = -(1, 0) = (-1, 0)$$

Thus :

$$f(x, y) = f(x(1, 0) + y(0, 1)) = xf(1, 0) + yf(0, 1)$$

$$f(x, y) = x(1, 2) + y(-1, 0) = (x - y, 2x)$$

Remark 6.5 If $\mathbb{R}^m$ and $\mathbb{R}^n$ are equipped with their canonical bases, then the linear map $f : \mathbb{R}^n \to \mathbb{R}^m$ associated with a matrix $A = (a_{ij})_{1 \le i \le m, 1 \le j \le n}$ is given by :

$$\forall (x_1, x_2, ..., x_n) \in \mathbb{R}^n, \quad f(x_1, x_2, ..., x_n) = A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Example 6.6 Consider the linear map :

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \quad A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$$

$$f(x, y) = A \begin{bmatrix} x \\ y \end{bmatrix} = (x - y, 2x)$$

Theorem 6.7 Let E, F, G be vector spaces over $\mathbb{K}$, equipped respectively with bases B, B', B'', and let $f : E \to F$ and $g : F \to G$ be linear maps. Then :

$$M_{(B,B'')}(g \circ f) = M_{(B',B'')}(g) M_{(B,B')}(f)$$

Remark 6.8 Consider the linear maps :

$$f : \mathbb{R}^3 \to \mathbb{R}^2, \quad g : \mathbb{R}^2 \to \mathbb{R}^2$$

$$(x, y, z) \mapsto (x + y + 2z, x - y), \quad (x, y) \mapsto (x - y, 2x + y)$$

If $\mathbb{R}^2$ and $\mathbb{R}^3$ are equipped with their canonical bases, then :

$$g \circ f : \mathbb{R}^3 \to \mathbb{R}^2, \quad (x, y, z) \mapsto g \circ f(x, y, z)$$

With :

$$M(g \circ f) = M(g) M(f)$$

$$M(g) = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \quad M(f) = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

$$M(g \circ f) = M(g) M(f) = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2 \\ 3 & 1 & 4 \end{bmatrix}$$

Thus :

$$g \circ f(x, y, z) = M(g \circ f) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = (2y + 2z, 3x + y + 4z)$$

Theorem 6.9 Let $f : E \to F$, where B is a basis of E and B' is a basis of F. We have :

$$f \text{ bijective } \iff \det M_{(B,B')}(f) \neq 0$$

Moreover, in this case :

$$M_{(B,B')}(f^{-1}) = (M_{(B,B')}(f))^{-1}$$

Example 6.10 Consider the linear map :

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y) \mapsto (x - y, x + y)$$

We show that f is bijective and calculate its inverse.
The matrix representation of f with respect to the basis B :

$$M_B(f) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = A$$

$$\det(A) = 2 \neq 0 \Rightarrow f \text{ is bijective.}$$

The inverse matrix is computed as :

$$(M_B(f))^{-1} = \frac{1}{\det(A)} C^T A$$

Where :

$$C_A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad C_A^T = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Thus :

$$(M_{(B,B')}(f))^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = M_B(f^{-1})$$

$$f^{-1}(x,y) = M_B(f^{-1}) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{x}{2} + \frac{y}{2} \\ -\frac{x}{5} + \frac{y}{2} \end{bmatrix}$$

Proposition 6.11 Let $A \in M(n,m)(\mathbb{K})$ be associated with the linear map $f : E \to F$ whose matrix representation with respect to bases B in E and B' in F is given by A. Then :

$$\text{rank}(A) = \text{rank}(f), \quad \text{rank}(A) = \text{rank}(A^T)$$