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Course of Algebra 2

Chapter 2

Linear Mappings

2.1 Definition and Examples

Definition 1. *Let E and F be two vector spaces over a field $\mathbb{K}$. A mapping $f : E \to F$ is said to be linear (or a homomorphism) if, $\forall x, y \in E, \forall \lambda \in \mathbb{K}$,*

- $f(x + y) = f(x) + f(y)$.
- $f(\lambda \cdot x) = \lambda \cdot f(x)$,

or equivalently:

$$f(\lambda x + \mu y) = \lambda f(x) + \mu f(y).$$

In other words, a linear mapping transforms the internal law of the starting space E into the internal law of the target space F, and similarly for the external law.

Remark 2.1.1. *We denote by $L(E; F)$ the set of linear mappings from E to F. If f and g are two linear mappings from E to F and $\lambda \in \mathbb{K}$ is a scalar, then the mappings $f + g$ and λf defined, for all $x \in E$, by*

$$(f + g)(x) = f(x) + g(x),$$

$$(\lambda f)(x) = \lambda f(x),$$

are also linear mappings from E to F. The space $(L(E; F), +, \times)$ is a $\mathbb{K}$-vector space.

Example 2.1.2.

- *Let the mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (-x, -y)$. Let us show that it is linear. In fact, let $\mathbf{u} = (x, y) \in \mathbb{R}^2$, $\mathbf{v} = (x_0, y_0) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$. Then we have:*

$$f(\mathbf{u} + \mathbf{v}) = f\left(\begin{pmatrix} x + x_0 \\ y + y_0 \end{pmatrix}\right) = \begin{pmatrix} -(x + x_0) \\ -(y + y_0) \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix} + \begin{pmatrix} -x_0 \\ -y_0 \end{pmatrix} = f(\mathbf{u}) + f(\mathbf{v}).$$

Next,

$$f(\lambda \mathbf{u}) = f\left(\begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix}\right) = \begin{pmatrix} -(\lambda x) \\ -(\lambda y) \end{pmatrix} = \lambda \begin{pmatrix} -x \\ -y \end{pmatrix} = \lambda f(\mathbf{u}).$$

- *Let f be the mapping defined from $\mathbb{R}^2$ to itself by:*

$$f(x, y) = (x - y, 0).$$

f is linear if and only if $\forall v = (x, y), u = (x_0, y_0) \in \mathbb{R}^2$ we have

$$f(\lambda v + \mu u) = \lambda f(v) + \mu f(u).$$

According to the operations defined on the vector space $\mathbb{R}^2$,

$$\lambda v + \mu u = (\lambda x + \mu x_0, \lambda y + \mu y_0).$$

Let $X = \lambda x + \mu x_0$ and $Y = \lambda y + \mu y_0$. Thus,

$$f(\lambda v + \mu u) = f(X, Y) = (X - Y, 0)$$

$$= (\lambda(x - y) + \mu(x_0 - y_0), 0) = \lambda(x - y, 0) + \mu(x_0 - y_0, 0)$$

$$= \lambda f(x, y) + \mu f(x_0, y_0) = \lambda f(v) + \mu f(u).$$

- *Let f be the mapping defined from $\mathbb{R}^3$ to itself by:*

$$f(x, y, z) = (-x + y + 2z, y - z).$$

f is linear if and only if $\forall v = (x, y, z), u = (x_0, y_0, z_0) \in \mathbb{R}^3$ we have

$$f(\lambda v + \mu u) = \lambda f(v) + \mu f(u).$$

We know that $\lambda v + \mu u = (\lambda x + \mu x_0, \lambda y + \mu y_0, \lambda z + \mu z_0)$.
Let $X = \lambda x + \mu x_0, Y = \lambda y + \mu y_0$ and $Z = \lambda z + \mu z_0$. Thus,

$$\begin{aligned} f(\lambda v + \mu u) = f(X, Y, Z) &= (-X + Y + 2Z, Y - Z) \\ &= (\lambda(-x + y + 2z) + \mu(-x_0 + y_0 + 2z_0), \lambda(y - z) - \mu(y_0 - z_0)) \\ &= \lambda(-x + y + 2z, y - z) + \mu(-x_0 + y_0 + 2z_0, y_0 - z_0) \\ &= \lambda f(x, y, z) + \mu f(x_0, y_0, z_0) = \lambda f(v) + \mu f(u). \end{aligned}$$

2.2 Some Properties

Definition 2.
- *A linear mapping from E to E is called an endomorphism of E. The vector space of endomorphisms of E is denoted by $L(E)$.*
- *A linear mapping from E to $\mathbb{K}$ is called a linear functional or linear form.*
- *A bijective linear mapping is called an isomorphism.*
- *Bijective endomorphisms of E are called automorphisms of E.*

Proposition 2.2.1. *Let f be a linear mapping from E to F.*

1. $f(O_E) = O_F$
2. $\forall x \in E, f(-x) = -f(x)$

Proof. We have,

1. $f(O_E) = f(O_E + O_E) = f(O_E) + f(O_E) \implies f(O_E) = O_F$
2. $f(-x) + f(x) = f(-x + x) = f(O_E) = O_F \implies f(-x) = -f(x)$

□

Proposition 2.2.2.
1. *The reciprocal mapping of a linear mapping is linear.*
2. *The composition of two linear mappings is a linear mapping.*

2.3 Kernel and Image of a Linear Mapping

Definition 3. *Let $f : E \to F$ be a linear mapping. The kernel of f, denoted $\ker f$, is the set*

$$\ker f := f^{-1}(0) = \{x \in E \mid f(x) = 0\},$$

and the image of f, denoted Imf, is the set

$$Imf := f(E) = \{f(x) \mid x \in E\}.$$

2.3.1 Examples

Determine the kernels and images of the mappings defined in the following examples:

1.
$$\ker(f) = \{(x,y) \in \mathbb{R}^2 : f(x,y) = \begin{pmatrix} -x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\} = \{0\}.$$

2.
$$\ker(g) = \{P \in \mathbb{R}^2[X] : g(P) = \begin{pmatrix} P(0) \\ P'(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\},$$

$P = aX^2 + bX + c$, thus
$$P(0) = P'(0) = 0 \Leftrightarrow c = b = 0.$$
This implies that,
$$P \in \ker(g) \Leftrightarrow P = aX^2,$$
concluding that
$$\ker(g) = \{P \in \mathbb{R}^2[X] : P = aX^2, a \in \mathbb{R}\} = \langle X^2 \rangle.$$

3.
$$\ker(T) = \{f \in E : T(f) = f' = 0\} = \{f \in E : f \text{ is constant}\}.$$

4.
$$\text{Im }(f) = \{f(x,y), (x,y) \in \mathbb{R}^2\} = \left\{\begin{pmatrix} -x \\ y \end{pmatrix}, (x,y) \in \mathbb{R}^2\right\} = \left\{x\begin{pmatrix} -1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix}, (x,y) \in \mathbb{R}^2\right\}.$$
Thus,
$$\text{Im }(f) = \text{span}\left\{\begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} = \mathbb{R}^2.$$

5.
$$\begin{aligned}
\text{Im }(g) &= \{g(P) \mid P = aX^2 + bX + c \in \mathbb{R}^2[X]\} \\
&= \left\{\begin{pmatrix} P(0) \\ P'(0) \end{pmatrix} \mid P = aX^2 + bX + c \in \mathbb{R}^2[X]\right\} \\
&= \left\{\begin{pmatrix} c \\ b \end{pmatrix} \mid c, b \in \mathbb{R}\right\} \\
&= \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} \\
&= \mathbb{R}^2.
\end{aligned}$$

Proposition 2.3.1. *Let E and F be two vector spaces over a field $\mathbb{K}$, and let $f : E \to F$ be a linear mapping.*

1. $\ker f$ *is a vector subspace of E, and Imf is a vector subspace of F.*
2. *f is injective* $\Longleftrightarrow \ker f = \{0_E\}$.
3. *f is surjective* $\Longleftrightarrow Imf = F$.

Definition 4 (The rank of a mapping). *A linear mapping $f : E \to F$ has a finite rank if its image subspace, $Im(f)$, has a finite dimension. The dimension of $Im(f)$ is called the rank of f and is denoted by $rank(f)$.*

Example 2.3.2. *Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by*
$$f(x,y,z) = (-x + y, x - z, y).$$
We have
$$Imf = \{f(x,y,z) \mid (x,y,z) \in \mathbb{R}^3\} = \{(-x + y, x - z, y) \mid (x,y,z) \in \mathbb{R}^3\}.$$

$$Imf = \{x(-1,1,0) + y(1,0,1) + z(0,-1,0) \mid (x,y,z) \in \mathbb{R}^3\}.$$

Thus, Imf is a subspace of $\mathbb{R}^3$ generated by $\{(-1,1,0),(1,0,1),(0,-1,0)\}$. It is easy to show that this set is linearly independent, and thus forms a basis of $\mathbb{R}^3$. Therefore,

$$\dim Imf = 3, \quad rank(f) = 3, \quad Imf = \mathbb{R}^3.$$

We conclude that f is surjective. Moreover, $\ker f = \{(x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = (0,0,0)\}$,

$$\implies \ker f = \{(x,y,z) \in \mathbb{R}^3 \mid (-x+y, x-z, y) = (0,0,0)\} \implies x = y = z = 0$$

Thus, $\ker f = \{(0,0,0)\}$, so f is also injective, then is bijective.

Theorem 2.3.3 (Rank Theorem). *Let f be a linear mapping from E to F with E of finite dimension. We have:*

$$\dim E = \dim \ker(f) + rank(f)$$

Proposition 2.3.4. *Let E and F be two vector spaces over a field $\mathbb{K}$, and let f and g be two linear mappings from E to F. If E is of finite dimension n and $\{e_1, e_2, \ldots, e_n\}$ is a basis of E, then*

$$\forall k \in \{1,2,\ldots,n\}, f(e_k) = g(e_k) \iff \forall x \in E, f(x) = g(x).$$

In other words, for two linear mappings f and g from E to F to be equal, it suffices for them to coincide on the basis of the $\mathbb{K}$-vector space E.

Proof. The implication $(\Leftarrow)$ is evident.

For $(\Rightarrow)$, E is generated by $\{e_1, e_2, \ldots, e_n\}$, so for all $x \in E$, there exist $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{K}$ such that

$$x = \lambda_1 e_1 + \lambda_2 e_2 + \ldots + \lambda_n e_n.$$

Since f and g are linear, we have

$$f(x) = f(\lambda_1 e_1 + \lambda_2 e_2 + \ldots + \lambda_n e_n) = \lambda_1 f(e_1) + \lambda_2 f(e_2) + \ldots + \lambda_n f(e_n),$$

$$g(x) = g(\lambda_1 e_1 + \lambda_2 e_2 + \ldots + \lambda_n e_n) = \lambda_1 g(e_1) + \lambda_2 g(e_2) + \ldots + \lambda_n g(e_n).$$

Thus, if we suppose that for all $k \in \{1,2,\ldots,n\}$, $f(e_k) = g(e_k)$, we deduce that for all $x \in E$,

$$f(x) = g(x).$$

□

Proposition 2.3.5. *Let $f : E \to F$ be a linear mapping between two $\mathbb{K}$-vector spaces of the same finite dimension. Then the following statements are equivalent:*

1. *f is injective.*
2. *f is surjective.*
3. *f is bijective.*
4. $\dim E = rankf$.
5. $\dim(Im(f)) = \dim(F) \iff Im(f) = F$.
6. $\dim \ker(f) = 0 \iff \ker(f) = \{0_E\}$.

Proof. Let start by

1) $\Rightarrow$ 2) If f is injective, we have $\dim \ker f = 0$, so $\mathrm{rank}(f) = \dim E - \dim \ker f = \dim E = \dim F$. Since $\mathrm{Im} f$ is a subspace of F, we have $\mathrm{Im} f = F$ and f is surjective.

2) $\Rightarrow$ 3) If f is surjective, the same calculation shows that it is injective and hence bijective.

3) $\Rightarrow$ 4) If f is bijective, then $\ker f = \{0\}$ and $\mathrm{Im} f = F$. Consequently, $\dim E = \dim \ker f + \mathrm{rank}(f) = \mathrm{rank}(f)$.

4) $\Rightarrow$ 1) If $\dim E = \mathrm{rank}(f)$, then according to the previous theorem, $\dim \ker f = 0$, which implies $\ker f = \{0\}$ and hence f is injective.

□