Mohamed Boudiaf University-M'sila
Department of Computer Science
First year in Computer Science
2025-2026

Course of Algebra 2

Chapter 1

Vector Spaces

1.1 Definition of Vector Space

Let K be a commutative field (generally $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and consider a non-empty set E equipped with:
- An internal composition law (binary operation) denoted by $(+)$:

$$(+) : E \times E \to E$$

$$(x, y) \mapsto x + y$$

- An external composition law (scalar multiplication) denoted by $(\cdot)$:

$$(\cdot) : K \times E \to E$$

$$(\lambda, x) \mapsto \lambda \cdot x$$

Definition 1. *A vector space over the field $\mathbb{K}$ or $\mathbb{K}$-vector space is a triplet $(E, +, \cdot)$ such that:*
1. $(E, +)$ is an abelian group with the neutral element denoted by 0_E, meaning:

- ***Associativity of** $(+)$:*

$$\forall x, y, z \in E, \quad (x + y) + z = x + (y + z)$$

- ***Commutativity of** $(+)$:*

$$\forall x, y \in E, \quad x + y = y + x$$

- ***Existence of a neutral element:***

$$\exists 0_E \in E, \quad \forall x \in E, \quad 0_E + x = x$$

- ***Existence of an additive inverse:***

$$\forall x \in E, \quad \exists x' \in E, \quad x + x' = 0_E$$

- **Distributivity of scalar multiplication over vector addition**:

$$\forall \lambda \in K, \quad \forall x, y \in E, \quad \lambda \cdot (x + y) = \lambda \cdot x + \lambda \cdot y$$

- **Distributivity of field addition over scalar multiplication**:

$$\forall \lambda, \nu \in K, \quad \forall x \in E, \quad (\lambda + \nu) \cdot x = \lambda \cdot x + \nu \cdot x$$

- **Associativity of scalar multiplication**:

$$\forall \lambda, \nu \in K, \quad \forall x \in E, \quad (\lambda \cdot \nu) \cdot x = \lambda \cdot (\nu \cdot x)$$

- **Existence of multiplicative identity**:

$$\forall x \in E, \quad 1_K \cdot x = x$$

The elements of E are called vectors, and the elements of K are called scalars.

Example 1.1.1. $(\mathbb{R}, +, \cdot)$ *is a vector space over* $\mathbb{R}$.

Example 1.1.2. $(\mathbb{C}, +, \cdot)$ *is a vector space over* $\mathbb{C}$.

Example 1.1.3. *Let* $K = \mathbb{R}$ *and consider* $E = \mathbb{R}^2$ *with two operations:*
- An internal composition law $(+)$*:*

$$\forall (x_1, x_2), (y_1, y_2) \in \mathbb{R}^2, \quad (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$

- An external composition law $(\cdot)$*:*

$$\forall \lambda \in \mathbb{R}, \quad \forall (x_1, x_2) \in \mathbb{R}^2, \quad \lambda \cdot (x_1, x_2) = (\lambda \cdot x_1, \lambda \cdot x_2)$$

$(\mathbb{R}^2, +, \cdot)$ *is a vector space over* $\mathbb{R}$. *The neutral element for the internal law is the zero vector* $0_{\mathbb{R}^2} = (0, 0)$ *and the additive inverse of* (x_1, x_2) *is* $(-x_1, -x_2) = -(x_1, x_2)$.

Example 1.1.4. *Let* $K = \mathbb{R}$ *and consider* $E = \mathbb{R}^n$ *with two operations:*
- An internal composition law $(+)$*:*

$$\forall (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n, \quad (x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

- An external composition law $(\cdot)$*:*

$$\forall \lambda \in \mathbb{R}, \quad \forall (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, \quad \lambda \cdot (x_1, x_2, \ldots, x_n) = (\lambda \cdot x_1, \lambda \cdot x_2, \ldots, \lambda \cdot x_n)$$

1.2 Examples and Propositions

$(\mathbb{R}^n, +, \cdot)$ is a vector space over $\mathbb{R}$. The neutral element for the internal law is the zero vector $0_{\mathbb{R}^n} = (0, 0, \ldots, 0)$ and the additive inverse of $(x_1, x_2, \ldots, x_n)$ is $(-x_1, -x_2, \ldots, -x_n) = -(x_1, x_2, \ldots, x_n)$.

Proposition 1.2.1. *If* E *is a* K*-vector space, then the following properties hold:*

1. $\forall x \in E, \quad 0 \cdot x = 0_E$
2. $\forall \lambda \in K, \quad \lambda \cdot (0_E) = 0_E$
3. $\forall x \in E, \quad \forall \lambda \in K, \quad \lambda \cdot x = 0_E \iff \lambda = 0$ *or* $x = 0_E$
4. $\forall x, y \in E, \quad \forall \lambda \in K, \quad \lambda \cdot (x - y) = \lambda \cdot x - \lambda \cdot y$
5. $-1_K \cdot x = -x$

1.3 Definition of Vector Subspace

Definition 2. *Let* E *be a* K*-vector space. A subset* F *of* E *is called a Vector Subspace if:*

1. $F \neq \emptyset$ *(i.e.,* $0_E \in F$*)*
2. $\forall x, y \in F, \quad x + y \in F$
3. $\forall x \in F, \quad \forall \lambda \in K, \quad \lambda \cdot x \in F$

Remark 1.3.1. 1. *The first condition means that the zero vector of* E *must also be in* F*. In fact, it is sufficient to prove that* F *is non-empty.*

2. *The second condition means that* F *is closed under addition.*

3. *The third condition means that* F *is closed under scalar multiplication.*

Theorem 1.3.2. *Let $(E,+,\cdot)$ be a K-vector space and let F be a non-empty subset of E. The following are equivalent:*

1. *F is a K-vector subspace.*
2. *F is closed under addition and scalar multiplication, that is:*

$$\forall x,y \in F, \quad \forall \lambda \in K, \quad x+y \in F \text{ and } \lambda \cdot x \in F$$

3. *$\forall x,y \in F, \quad \forall \lambda,\nu \in K, \quad \lambda x+\nu y \in F$, therefore:*

$$F \text{ is a vector subspace} \iff \begin{cases} F \neq \emptyset \\ \forall x,y \in F, \quad \forall \lambda,\nu \in K, \quad \lambda x+\nu y \in F \end{cases}$$

4. *$\forall x,y \in F, \quad \forall \lambda,\nu \in K, \quad \lambda x+\nu y \in F$, therefore:*

$$F \text{ is a vector subspace} \iff \begin{cases} 0_E \in F \\ \forall x,y \in F, \quad \forall \lambda,\nu \in K, \quad \lambda x+\nu y \in F \end{cases}$$

Example 1.3.3. *$F_1 = \{(x,y) \in \mathbb{R}^2 \mid x+y=0\}$ is a vector subspace of $\mathbb{R}^2$ because:*

1. *$0_E = 0_{\mathbb{R}^2} = (0,0) \in F_1 \implies F_1 \neq \emptyset$*
2. *$\forall (x_1,x_2),(y_1,y_2) \in F_1, \quad \forall \lambda,\nu \in \mathbb{R}$, we show that $\lambda(x_1,x_2)+\nu(y_1,y_2) \in F_1$, that is, $(\lambda x_1+\nu y_1, \lambda x_2+\nu y_2) \in F_1$. We have:*

$$\lambda x_1+\nu y_1+\lambda y_2+\nu x_2 = \lambda(x_1+y_1)+\nu(x_2+y_2) = \lambda \cdot 0+\nu \cdot 0 = 0$$

because $(x_1,x_2) \in F_1 \implies x_1+x_2=0$ and $(y_1,y_2) \in F_1 \implies y_1+y_2=0$. Thus, $\lambda(x_1,x_2)+\nu(y_1,y_2) \in F_1$, therefore F_1 is a vector subspace.

Example 1.3.4. *$F_2 = \{(x,y) \in \mathbb{R}^2 \mid x+y=2\}$ is not a vector subspace of $\mathbb{R}^2$ because $(0,0) \notin F_2$.*

Theorem 1.3.5. *The intersection of a non-empty family of vector subspaces is a vector subspace.*

Remark 1.3.6. *The union of two vector subspaces is not necessarily a vector subspace.*

Example 1.3.7. *Let $F_1 = \{(0,y) \in \mathbb{R}^2\}$ and $F_2 = \{(x,0) \in \mathbb{R}^2\}$ be two vector subspaces. We have $F_1 \cup F_2$ is not a vector subspace because $u=(1,0) \in F_1$ and $v=(0,1) \in F_2$, but $u+v=(1,1) \notin F_1 \cup F_2$ since $(1,1) \notin F_1$ and $(1,1) \notin F_2$.*

1.4 Linear Combinations

Definition 3. *Let $n > 1$ be an integer, and let $v_1,v_2,\ldots,v_n$ be n vectors in a K-vector space E. Any vector of the form*

$$u = \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_n v_n$$

(where $\lambda_1,\lambda_2,\ldots,\lambda_n$ are elements of K) is called a linear combination of the vectors $v_1,v_2,\ldots,v_n$. The scalars $\lambda_1,\lambda_2,\ldots,\lambda_n$ are called the coefficients of the linear combination.

Remark 1.4.1. *If $n=1$, then $u=\lambda_1 v_1$, and we say that u is collinear with v_1.*

Example 1.4.2. *1. In the $\mathbb{R}$-vector space $\mathbb{R}^3$, $(3,3,1)$ is a linear combination of the vectors $(1,1,0)$ and $(1,1,1)$ because we have the equality*

$$(3,3,1) = 2(1,1,0)+(1,1,1).$$

2. In the $\mathbb{R}$-vector space $\mathbb{R}^2$, the vector $u=(2,1)$ is not collinear with the vector $v_1=(1,1)$ because if it were, there would exist a real number λ such that $u=\lambda v_1$, which would be equivalent to the equality

$$(2,1) = (\lambda,\lambda).$$

1.5 Linearly Independent Families

Definition 4. *A family $\{v_1, v_2, \ldots, v_n\}$ of E is a linearly independent family if:*

$$\forall \lambda_1, \lambda_2, \ldots, \lambda_n \in K, \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_n v_n = 0_E \implies \lambda_1 = \lambda_2 = \ldots = \lambda_n = 0.$$

Otherwise, if there exists a non-zero linear combination, the family is said to be linearly dependent.

Example 1.5.1. *Show that $v_1 = (1, 0, 1)$, $v_2 = (0, 2, 2)$, and $v_3 = (3, 7, 1)$ are linearly independent in $\mathbb{R}^3$.*
Let $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$, we have

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = (0, 0, 0) \implies \lambda_1(1, 0, 1) + \lambda_2(0, 2, 2) + \lambda_3(3, 7, 1) = (0, 0, 0)$$

which gives us the system of equations:

$$\begin{cases} \lambda_1 + 3\lambda_3 = 0 \\ 2\lambda_2 + 7\lambda_3 = 0 \\ \lambda_1 + 2\lambda_2 + \lambda_3 = 0 \end{cases}$$

Solving this system, we get:

$$\begin{cases} \lambda_1 = -3\lambda_3 \\ \lambda_2 = -\frac{7}{2}\lambda_3 \\ \lambda_1 + 2\lambda_2 + \lambda_3 = 0 \end{cases}$$

Substituting the values of λ_1 and λ_2, we get:

$$-3\lambda_3 + 2\left(-\frac{7}{2}\lambda_3\right) + \lambda_3 = 0$$

$$-3\lambda_3 - 7\lambda_3 + \lambda_3 = 0$$

$$-9\lambda_3 = 0 \implies \lambda_3 = 0$$

Hence,

$$\lambda_1 = 0$$

$$\lambda_2 = 0$$

$$\lambda_3 = 0$$

Thus, the family $\{v_1, v_2, v_3\}$ is linearly independent.

1.6 Linearly Dependent Families

Definition 5. *A family $\{v_1, v_2, \ldots, v_n\}$ is said to be linearly dependent if it is not linearly independent, that is,*

$$\exists \lambda_1, \lambda_2, \ldots, \lambda_n \in K, \textit{not all zero, such that } \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_n v_n = 0_E.$$

Example 1.6.1. *Let $e_1 = (1, 2)$, $e_2 = (1, 1)$, and $e_3 = (0, 1) \in K^2$. The family $\{e_1, e_2, e_3\}$ is linearly dependent because*

$$e_1 - e_2 - e_3 = 0.$$

1.7 Span

Let $\{v_1, v_2, \ldots, v_n\}$ be a finite set of vectors in a vector space E. The set $\langle v_1, v_2, \ldots, v_n\rangle$, also denoted as $\text{span}(v_1, v_2, \ldots, v_n)$, is the set of all linear combinations of the vectors $\{v_1, v_2, \ldots, v_n\}$ and is called the span of $v_1, v_2, \ldots, v_n$. In other words,

$$u \in \langle v_1, v_2, \ldots, v_n\rangle \iff \exists \lambda_1, \lambda_2, \ldots, \lambda_n \in K \text{ such that } u = \sum_{k=1}^{n} \lambda_k v_k.$$

Example 1.7.1. *Let $u = (1,1,1)$ and $v = (1,2,3)$ be two vectors in $\mathbb{R}^3$. Determine $\langle u, v\rangle$.*
We have

$$(x,y,z) \in \langle u, v\rangle \iff (x,y,z) = \lambda u + \nu v$$

$$\iff (x,y,z) = \lambda(1,1,1) + \nu(1,2,3)$$

$$\iff (x,y,z) = (\lambda+\nu, \lambda+2\nu, \lambda+3\nu)$$

which gives us the system of equations:

$$\begin{cases} x = \lambda + \nu \\ y = \lambda + 2\nu \\ z = \lambda + 3\nu \end{cases}$$

Solving this system, we get:

$$\begin{cases} \lambda = y - x \\ \nu = 3x - 3y + z \end{cases}$$

Proposition 1.7.2. *Let A be a non-empty subset of $\mathbb{R}^n$. We denote*

$$\textit{Vect}(A) = \textit{the set of all linear combinations of elements of } A.$$

Then we have:

1. *Vect(A) is a subspace of $\mathbb{R}^n$,*
2. *Vect(A) is the smallest subspace of $\mathbb{R}^n$ containing A, in the sense that if a subspace E contains A, then it contains Vect(A). (If $A \subseteq E$, then Vect(A) $\subseteq E$ if E is a subspace.)*

Vect(A) is called the subspace generated by A.

1.8 Generating Set

Definition 6. *Let E be a K-vector space. A family $\{v_1, v_2, \ldots, v_n\}$ of vectors in E is said to be a generating set if:*

$$\forall v \in E, \exists \lambda_1, \lambda_2, \ldots, \lambda_n \in K \text{ such that } v = \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_n v_n.$$

In other words, the family $\{v_1, v_2, \ldots, v_n\}$ generates the vector space E.

Remark 1.8.1. *The vectors $\{v_1, v_2, \ldots, v_n\}$ form a generating set of E if and only if:*

$$\langle v_1, v_2, \ldots, v_n\rangle = E.$$

Example 1.8.2. *Consider the vectors $v_1 = (1,0,0)$, $v_2 = (0,1,0)$, and $v_3 = (0,0,1)$ in $\mathbb{R}^3$. The family $\{v_1, v_2, v_3\}$ is a generating set because any vector $v = (x,y,z)$ in $\mathbb{R}^3$ can be written as*

$$(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1).$$

1.9 Basis of a Generating Set

Definition 7. *Let E be a K-vector space. A family $\{v_1, v_2, \ldots, v_n\}$ of vectors in E is a basis of E if $\{v_1, v_2, \ldots, v_n\}$ is a linearly independent and generating set.*

Theorem 1.9.1. *Let $\{v_1, v_2, \ldots, v_n\}$ be a basis of the vector space E. Every vector v in E can be uniquely written as a linear combination of $\{v_1, v_2, \ldots, v_n\}$, that is,*

$$v = \sum_{k=1}^{n} \lambda_k v_k.$$

1.10 Finite Dimensional Vector Spaces

Definition 8. *A vector space is said to be finite-dimensional if and only if it is generated by a finite number of vectors.*

1.10.1 Dimension

Definition 9. *The dimension of a finite-dimensional vector space E is defined as the number of vectors in its basis and is denoted as $\dim(E)$.*

Proposition 1.10.1. *If E is a finite-dimensional vector space, then every subspace of E is also finite-dimensional.*

Remark 1.10.2. *If $\dim(E) = n$, to show that a family of n elements is a basis of E, it is sufficient to show that it is linearly independent or generating.*

Proposition 1.10.3. *Let E be a K-vector space of dimension n and F a subspace of E:*

1. $\dim(F) \leq \dim(E)$.
2. $\dim(F) = \dim(E) \iff E = F$.

Remark 1.10.4. *The vectors of K^n: $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, \ldots, 0)$, $\ldots$, $e_n = (0, 0, \ldots, 1)$ form a basis of K^n, called the canonical basis of K^n.*

Example 1.10.5. *Consider the family $B = \{(1, 2, 3), (2, 9, 0), (3, 3, 4)\}$ in $\mathbb{R}^3$. To show that B is a basis of $\mathbb{R}^3$, we know that $\dim(\mathbb{R}^3) = 3$ and since $|B| = 3$, it is sufficient to demonstrate that the elements of B are linearly independent.*

For all $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, if

$$\lambda_1(1, 2, 3) + \lambda_2(2, 9, 0) + \lambda_3(3, 3, 4) = 0_{\mathbb{R}^3},$$

then $\lambda_1 = \lambda_2 = \lambda_3 = 0$.
Suppose $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, and

$$\lambda_1(1, 2, 3) + \lambda_2(2, 9, 0) + \lambda_3(3, 3, 4) = 0_{\mathbb{R}^3},$$

which gives us the system of equations:

$$\begin{cases} \lambda_1 + 2\lambda_2 + 3\lambda_3 = 0 \\ 2\lambda_1 + 9\lambda_2 + 3\lambda_3 = 0 \\ 3\lambda_1 + 4\lambda_3 = 0 \end{cases}$$

Solving this system, we find $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Consequently, B is a basis of $\mathbb{R}^3$.

document

Theorem 1.10.6. *In a finite-dimensional K-vector space E, all bases have the same number of elements. This number is called the dimension of E and is denoted as $\dim(E)$.*

Remark 1.10.7. *By convention, a vector space reduced to $\{0\}$ is said to have dimension zero.*

Theorem 1.10.8. *Let E be a finite-dimensional K-vector space of dimension n. Then:*

1. *Every generating set containing only one non-zero element is a basis.*

2. *If $\{x_1, x_2, \dots, x_p\}$ is a generating set of elements of E, then $p \geq n$.*
   - *If $p = n$, then $\{x_1, x_2, \dots, x_p\}$ is a basis of E.*
   - *If $p > n$, then a basis of E can be extracted.*
3. *If $\{x_1, x_2, \dots, x_p\}$ is a linearly independent set of elements of E, then $p \leq n$.*
   - *If $p = n$, then $\{x_1, x_2, \dots, x_p\}$ is a basis of E.*
   - *If $p < n$, then it can be completed to a basis of E (Incomplete Basis Theorem).*
4. *If F is a subspace of E, then F is finite-dimensional and $\dim(F) \leq \dim(E)$. Moreover, F equals E if and only if $\dim(F) = \dim(E)$.*

1.11 Supplemental Vector Subspace

1.11.1 Sum of Two Vector Subspaces

Definition 10. *Let F_1 and F_2 be two subspaces of E. The sum of the subspaces F_1 and F_2, denoted $F_1 + F_2$, is defined by:*

$$F_1 + F_2 = \{x_1 + x_2 \mid x_1 \in F_1 \text{ and } x_2 \in F_2\}.$$

1.11.2 Direct Sum and Supplemental Subspaces

Definition 11. *The direct sum, denoted $F_1 \oplus F_2$, is defined as:*

$$F_1 \oplus F_2 = F_1 + F_2 \text{ and } F_1 \cap F_2 = \{0_E\}.$$

Definition 12. *Two subspaces F_1 and F_2 of a K-vector space E are supplemental subspaces of E if their direct sum equals the entire vector space E:*

$$E = F_1 \oplus F_2.$$

Proposition 1.11.1. *Let E be a finite-dimensional K-vector space of dimension n.*

1. *Every subspace F_1 has at least one supplemental subspace, that is, there exists a subspace F_2 such that $E = F_1 \oplus F_2$.*
2. *Let $F_1 \neq \emptyset$ and $F_2 \neq \emptyset$ be two subspaces of E and let B_1 be a basis of F_1 and B_2 a basis of F_2. The family $\{B_1, B_2\}$ is a basis if and only if $E = F_1 \oplus F_2$.*

Corollary 1.11.2. *Let E be a finite-dimensional K-vector space.*

$$E = F_1 \oplus F_2 \iff \begin{cases} F_1 \cap F_2 = \{0_E\} \\ \dim(E) = \dim(F_1) + \dim(F_2) \end{cases}$$

Proposition 1.11.3 (Grassmann Formula). *Let E be a finite-dimensional K-vector space and F_1 and F_2 be two vector subspaces of E, then*

$$\dim(F_1 + F_2) = \dim(F_1) + \dim(F_2) - \dim(F_1 \cap F_2).$$