Machine Structure 1

03 – Boolean Algebra

Introduction

General introduction:

Every computer is designed from integrated circuits which all have a specialized function (ALU, memory, decoding circuit instructions etc.)

These circuits are made from logical circuits, that the purpose is to execute operations on logical (binary) variables

Logic circuits are elaborated from electronic components (transistors, diodes, capacitors, etc.)

Types of logic circuits:

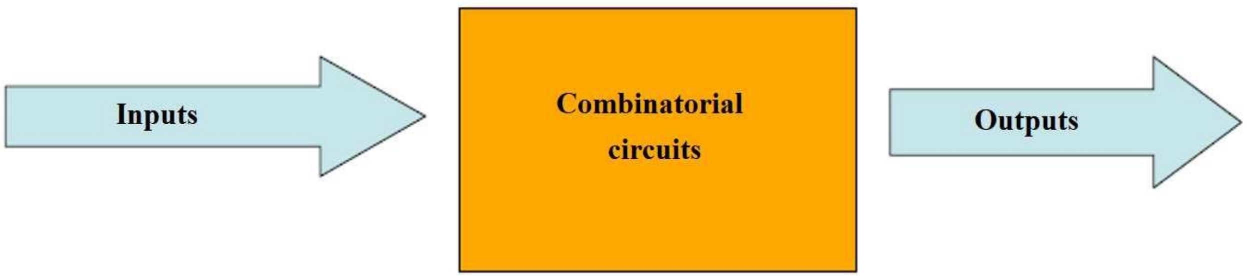
- Combinatorial
- Sequential

Introduction

Integrated circuit: set of electronic components (transistors, diodes, capacitors,…) which allows the creation of logic circuits

Logic Circuit: Set of logic functions where each one is defined by the inputs and outputs of the circuit, so each state of the outputs depending on the inputs must be specified

The functions of a circuit are developed using a concept called Boolean Algebra which represents the theoretical support of combinatorial circuits



Boolean Algebra

Definition:

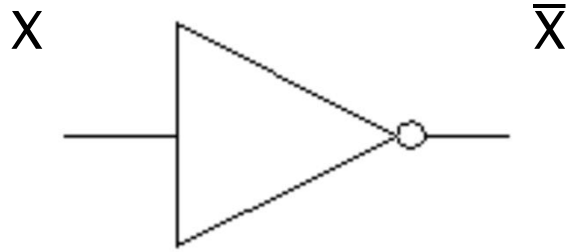
Set of rules and theorems that deals with two-state variables (0 or 1, True or False, closed or open) called Booleans using a set of basic logical operations: « NOT », « AND » And « OR »,

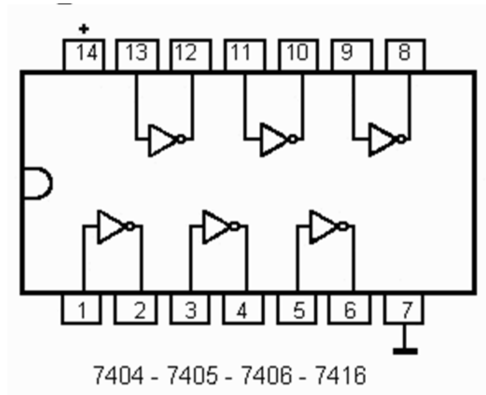
The truth table: is a table that allows to visualize the output values of a given function from the input variables that make up this function

Basic logical operations (gates):

- NOT: let X be a boolean variable, its negation, called complement, is noted $\overline{X}$ whose value is the inverse of the original value of X.

X	$\overline{X}$
0	1
1	0



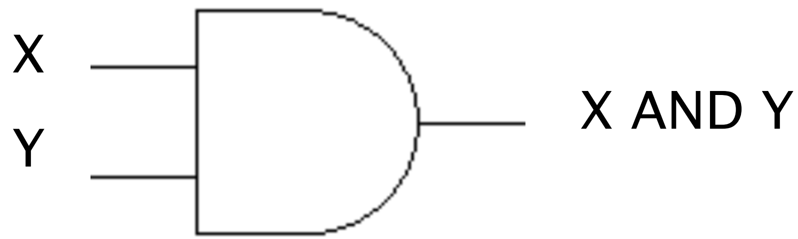


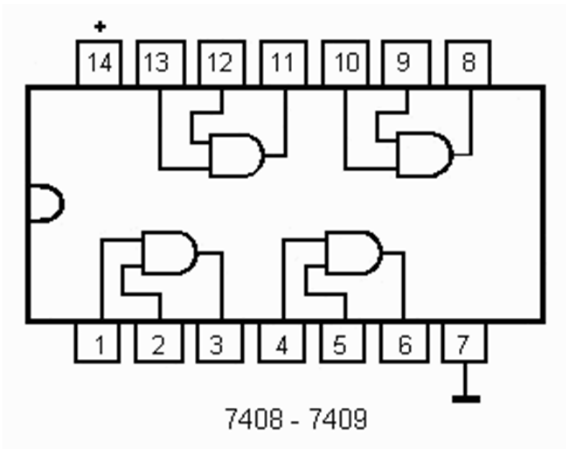
Boolean Algebra

Basic logical operations:

- AND: let X and Y two boolean variables, the value of (X AND Y) (also noted X . Y or XY) gives
  1 if X=1 and Y=1
  0 otherwise

X	Y	X AND Y
0	0	0
0	1	0
1	0	0
1	1	1





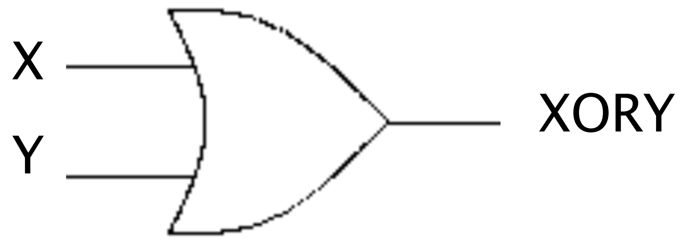
Boolean Algebra

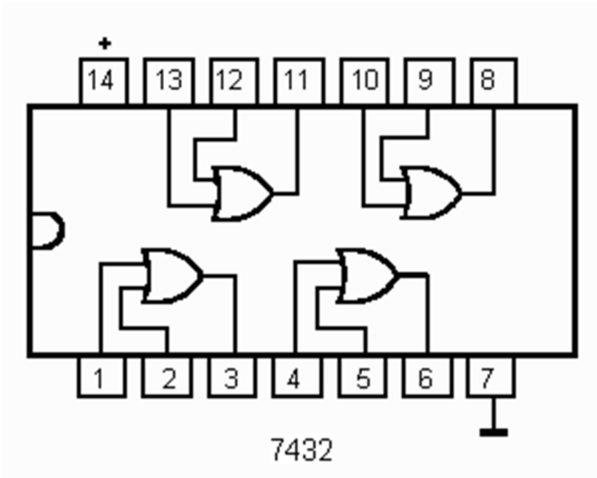
Basic logical operations:

- OR: let X and Y be two logical variables, the value of (X OR Y) (also noted (X+Y)) gives 1 if X=0 and Y=0

  2 otherwise

X	Y	X OR Y
0	0	0
0	1	1
1	0	1
1	1	1





Boolean Algebra

Logical function: is a combination of logical variables connected by logical operators (AND, OR and NOT)

Example: let a, b, c be three boolean variables

Here are some examples of logical functions:

$F1 = ab + \overline{a}b$, $F2 = (a+b)(\overline{a}b+b\overline{c})$

$F3 = (ab+c)(a\overline{b}c+a\overline{b}c)(ad+\overline{a}b\overline{c})$

The value of a logical function is equal to 0 or 1 depending on the values of the variables constituting the function, to find the possible values we have to draw up the truth table of this function,

The number of possible values = 2^n where n is the number of variables

Example: for F1 we have 4 possible values, F2: 8 possible values, and F3: 16 possible values,

Boolean Algebra

Truth table:

F1 $= ab + \bar{a}b$: 2 variables

=> 4 possible values

a	b	ab	$\bar{a}b$	F1
0	0	0	0	0
0	1	0	1	1
1	0	0	0	0
1	1	1	0	1

F2 $= (a+b)(\bar{a}b+b\bar{c})$ 3 variables => 8 values

a	b	c	$\bar{a}b$	$b\bar{c}$	$\bar{a}b+b\bar{c}$	a+b	F1
0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0
0	1	0	1	1	1	1	1
0	1	1	1	0	1	1	1
1	0	0	0	0	0	1	0
1	0	1	0	0	0	1	0
1	1	0	0	1	1	1	1
1	1	1	0	0	0	1	0

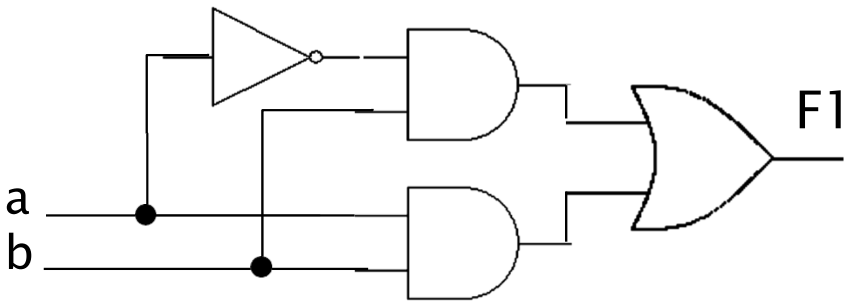
Boolean Algebra

Logical diagram of functions:

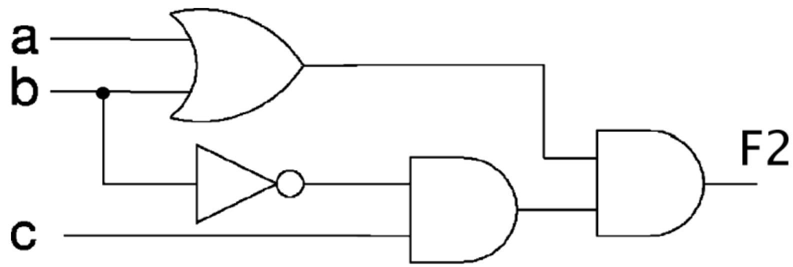
The function logic diagram is obtained by replacing each operator with the corresponding symbol

Example :

$$F1 = ab + \bar{a}b$$



$$F2 = (a+b)(\bar{b}.c)$$



Boolean Algebra

Fundamental laws:

$(A.B).C = A.(B.C) = A.B.C$	*Associativity*
$A.B = B.A$	*Commutativity*
$A.A = A$	*Identical*
$A.1 = A$	*Neutral element*
$A.0 = 0$	*Absorbing element*
$(A+B)+C = A+(B+C) = A+B+C$	*Associativity*
$A+B = B+A$	*Commutativity*
$A+A = A$	*Identical*
$A+0 = A$	*Neutral element*
$A+1 = 1$	*Absorbing element*

Boolean Algebra

Fundamental laws:

$$A.(B+C)=(A.B)+(A.C)$$ Product distribution

$$A+(B.C)=(A+B).(A+C)$$ Sum distribution

Axioms

$$1\text{-}\ (A.B)+(A.\overline{B})=A$$

$$2\text{-}\ A+\overline{A}B=A+B$$

$$3\text{-}\ A(\overline{A}+B)=AB$$

$$4\text{-}\ AB+\overline{A}C+BC=AB+\overline{A}C$$

$$5\text{-}\ (A+B)(\overline{A}+C)(B+C)=(A+B)(\overline{A}+C)$$

$$6\text{-}\ AB+A\overline{B}C=AB+AC$$

$$7\text{-}\ (A+B)(A+\overline{B}+C)=(A+B)(A+C)$$

$$A+(A.B)=A$$

$$A.(A+B)=A$$

$$(A+B).(A+\overline{B})=A$$

$$\overline{\overline{A}}=A$$

$$\overline{A}+A=1$$

$$\overline{A}.A=0$$

Boolean Algebra

Duality of Boolean algebra

Any logical equation expression remains true, if we replace the AND by OR, the OR by AND, the 1 by 0 and the 0 by 1.

Examples:

$$A + 1 = 1 \rightarrow A . 0 = 0$$

$$A + \overline{A} = 1 \rightarrow A . \overline{A} = 0$$

MORGANE's Theorem:

- $\overline{A . B} = \overline{A} + \overline{B}$
- $\overline{A + B} = \overline{A} . \overline{B}$

In general :

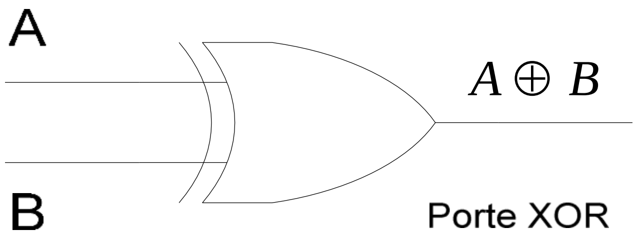
- $\overline{A.B.C. \ldots} = \overline{A} + \overline{B} + \overline{C} + \ldots$
- $\overline{A + B + C + \ldots} = \overline{A}.\overline{B}.\overline{C} \ldots$

Boolean Algebra

Other logical operators:

- The XOR exclusive-OR:

$$A \oplus B = \overline{A}.B + A.\overline{B}$$



A	B	A ⊕ B
0	0	0
0	1	1
1	0	1
1	1	0

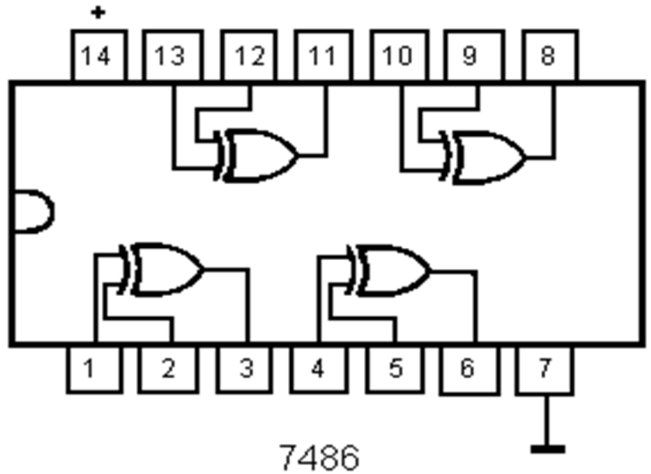
Other forms of XOR:

$$A \oplus B = (A + B) \bullet \overline{(A \bullet B)}$$

$$A \oplus B = (A \bullet \overline{B}) + (B \bullet \overline{A})$$

$$A \oplus B = \overline{(A \bullet B) + (\overline{A} \bullet \overline{B})}$$

$$A \oplus B = (A + B) \bullet (\overline{A} + \overline{B})$$

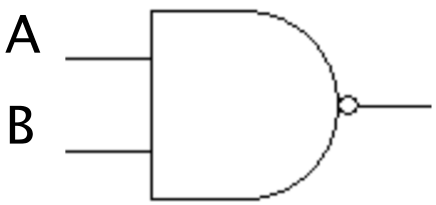


Boolean Algebra

Other logical operators:

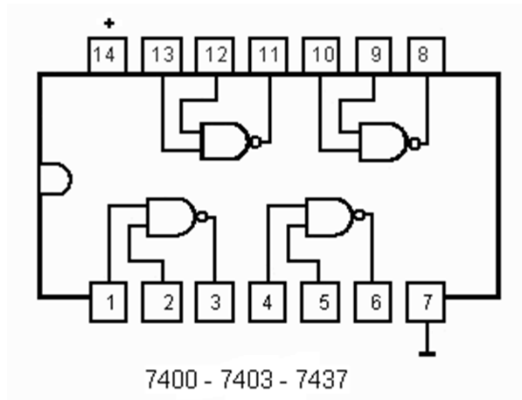
- The NO-AND: (NAND)

$$Y = \overline{A.B}$$



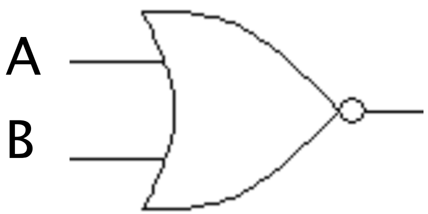
$$Y = \overline{A.B}$$

A	B	$Y = \overline{A.B}$
0	0	1
0	1	1
1	0	1
1	1	0



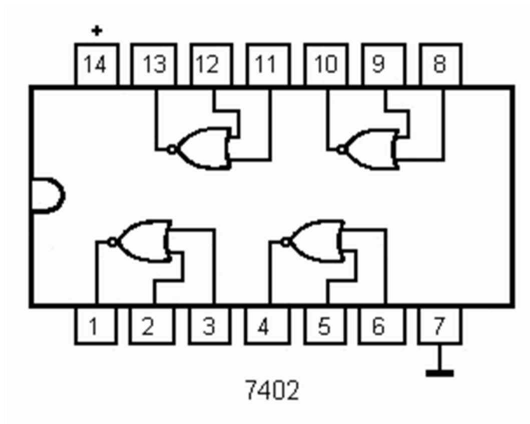
- The NO-OR: (NOR)

$$Y = \overline{A + B}$$



$$Y = \overline{A + B}$$

A	B	$Y = \overline{A + B}$
0	0	1
0	1	0
1	0	0
1	1	0



Boolean Algebra

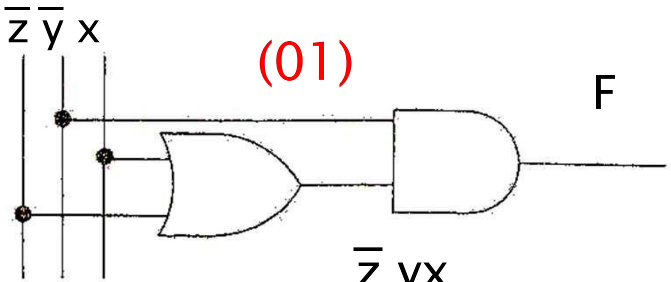
Using different gates to express logic functions:

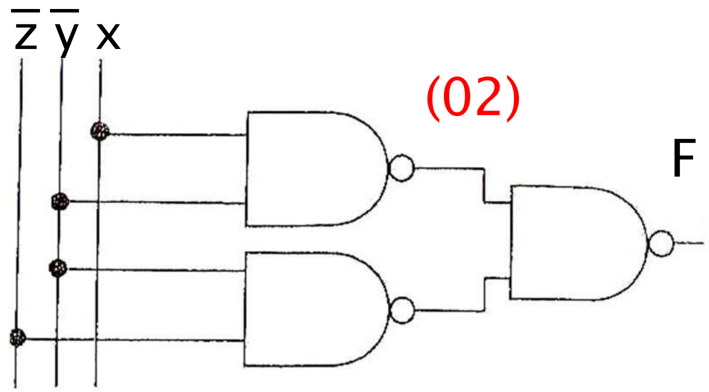
The goal is to transform a function according to the available integrated circuits with the corresponding gates.

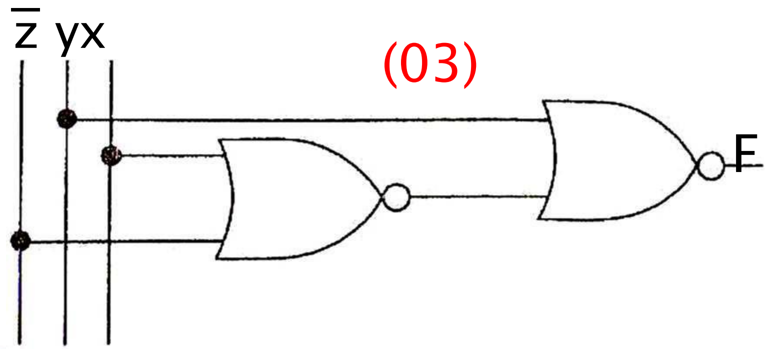
Example: let the function F $= \bar{y}(x+\bar{z})$ … (01) using AND And OR

$F = \overline{\overline{\bar{y}(x+\bar{z})}} = \overline{\overline{\bar{y}x+\bar{y}\bar{z}}} = \overline{\overline{\bar{y}x}.\overline{\bar{y}\bar{z}}}$ …(02) using NAND

$F = \overline{\overline{\bar{y}(x+\bar{z})}} = \overline{y+\overline{(x+\bar{z})}}$ …(03) using NOR







Boolean Algebra

Simplification of logic functions:

It consists to write a function by using a minimum number of terms and logical operators. The axioms of Boolean algebra cited above must be used.

Example 1: simplify, if possible, the function F:

$F = (a+b)(\overline{b}+c)(\overline{a}+c)$ We have three terms and 5 operators

$F = (a\overline{b}+ac+b\overline{b}+bc)(a+c) = (a\overline{b}+ac+bc)(\overline{a}+c)$

$F = (\overline{a}a\overline{b}+\overline{a}ac+\overline{a}bc+a\overline{b}c+acc+bcc)/$

$F = (\overline{a}bc+a\overline{b}c+ac+bc) = (bc(\overline{a}+1)+ac(\overline{b}+1))$

$F = bc+ac = c(a+b)$ We have two terms and 2 operators

Example2:simplify, if possible, the function F:

$\mathrm{F} = (\mathrm{a}+\mathrm{b})(\overline{\mathrm{a}}+\overline{\mathrm{b}}+\overline{\mathrm{c}})(\mathrm{a}+\mathrm{c})$ (to do)

Boolean Algebra

The Maxterms and Minterms in the truth table:

Minterm: for each value of the function equal to 1 we take the product of the variables constituting the function, leaving the variable as it is if its value is 1, otherwise we take its complement,

Maxterm: for each value of the function equal to 0 we take the sum of the variables constituting the function, leaving the variable as it is if its value is 0, otherwise we take its complement,

Boolean Algebra

The Max terms and Min terms in the truth table:

Example :

A	B	C	F		
0	0	0	0	$\longrightarrow$	$A + B + C$: max term
0	0	1	0	$\longrightarrow$	$A + B + \overline{C}$: max term
0	1	0	0	$\longrightarrow$	$A + \overline{B} + C$: max term
0	1	1	1	$\longrightarrow$	$\overline{A} . B . C$: min term
1	0	0	0	$\longrightarrow$	$\overline{A} + B + C$: max term
1	0	1	1	$\longrightarrow$	$A . \overline{B} . C$: min term
1	1	0	1	$\longrightarrow$	$A . B . \overline{C}$: min term
1	1	1	1	$\longrightarrow$	$A . B . C$: min term

Boolean Algebra

The Maxterms and Minterms in the truth table:

From the truth table, we can define the function F:

Sum of products (using Minterms)

$$F = \overline{A}.B.C + A.\overline{B}.C + A.B.\overline{C} + A.B.C$$

Product of sums (using Maxterms)

$$F = (A + B + C)(A + B + \overline{C})(A + \overline{B} + C)(\overline{A} + B + C)$$

Boolean Algebra

The Maxterms and Minterms in the truth table:

We can do the reverse to obtain the truth table from a function that is written in the form of products of sums or in the form of sum of products,

Example1: The Minterm $\overline{A}.B.C$ can be formed as: 0 1 1 for A, B and C respectively, so the value of F for A=0, B = 1 and C = 1 is 1

Example2: The Maxterm $(A+\overline{B}+C)$ can be formed as: 0 1 0 for A, B and C respectively, so the value of F for A=0, B = 1 and C = 0 is 0

Boolean Algebra

Canonical Forms:

A canonical form is a representation in which each term includes all the variables that make up the function.

Example : $F(A, B, C) = AB\overline{C} + A\overline{C}B + \overline{A}BC$

There are 04 canonical forms:

1) First canonical form
2) Second canonical form
3) Third canonical form
4) Fourt canonical form

Boolean Algebra

1) First canonical form (disjunctive form)

Represents the sum of the products form (Sum of the Minterms)

Example : $F(A,B,C) = \overline{A}.B.C + A.\overline{B}.C + A.B.\overline{C} + A.B.C$

A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

Reduced representation using decimal degits

03 variables => 03 bits

On three bits: $(011)_2 = (3)_{10}$, $(101)_2 = (5)_{10}$, $(110)_2 = (6)_{10}$ and $(111)_2 = (7)_{10}$

$F = R(3,5,6,7) = \Sigma(3,5,6,7)$

Boolean Algebra

2) Second canonical form (conjunctive form)

Represents the product of the sums form (product of the Maxterms)

Example : $F(A,B,C) = (A+B+C)(A+B+\overline{C})(A+\overline{B}+C)(\overline{A}+B+C)$

A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

Reduced representation using decimal degits

03 variables => 03 bits

On three bits: $(000)_2 = (0)_{10}$, $(001)_2 = (1)_{10}$, $(010)_2 = (2)_{10}$ and $(100)_2 = (4)_{10}$

F = P(0,1,2,4) = $\Pi(0,1,2,4)$

Boolean Algebra

Equivalence between 1st and 2nd canonical forms

The 1st and 2nd canonical forms are equivalent => two representations of the same function

Example: Let F be defined by the following truth table

$$F(A,B) = \overline{A}.B + A.\overline{B} + A.B$$ 1st canonical form

$$= (A + B)$$ 2nd canonical form

A	B	F
0	0	0
0	1	1
1	0	1
1	1	1

$$\overline{A}.B + A.\overline{B} + A.B = (\overline{A} + A)(\overline{A} + \overline{B})(A + B)(\overline{B} + B) + AB$$

$$= (\overline{A} + \overline{B})(A + B) + AB$$

$$= (A + \overline{A} + \overline{B})(A + A + B)(B + \overline{A} + \overline{B})(B + A + B)$$

$$= (A + B)(A + B)$$

$$= (A + B)$$

Boolean Algebra

3) Third canonical form

It is deduced from the 1st canonical form, it consists of representing a function by using only NAND gates

Example : $F(A,B,C) = \overline{A}.B.C + A.\overline{B}.C + A.B.\overline{C} + A.B.C$

$$= \overline{\overline{\overline{A}.B.C + A.\overline{B}.C + A.B.\overline{C} + A.B.C}}$$

$$= \overline{(\overline{\overline{A}.B.C})(\overline{A.\overline{B}.C})(\overline{A.B.\overline{C}})(\overline{A.B.C})}$$

4) Fourth canonical form

It is deduced from the 2nd canonical form, it consists of representing a function by using only the NOR gates

Example : $F(A,B,C) = (A+B+C)(A+B+\overline{C})(A+\overline{B}+C)(\overline{A}+B+C)$

$$= \overline{\overline{(A+B+C)(A+B+\overline{C})(A+\overline{B}+C)(\overline{A}+B+C)}}$$

$$= \overline{\overline{(A+B+C)} + \overline{(A+B+\overline{C})} + \overline{(A+\overline{B}+C)} + \overline{(\overline{A}+B+C)}}$$

Boolean Algebra

Examples of canonical forms:

1) Find the fourth canonical forms of the function F defined by the following truth table:

x	y	z	F
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

1st: $F = \overline{x}.\overline{y}.z + \overline{x}.y.\overline{z} + x.\overline{y}.\overline{z} + x.\overline{y}.z = \sum(1,2,4,5)$

2nd: $F = (x+y+z)(x+\overline{y}+\overline{z})(\overline{x}+\overline{y}+z)(\overline{x}+\overline{y}+\overline{z})$

$F = \prod(0,3,6,7)$

3rd: $F = \overline{(\overline{\overline{x}.\overline{y}.z})(\overline{\overline{x}.y.\overline{z}})(\overline{x.\overline{y}.\overline{z}})(\overline{x.\overline{y}.z})}$

4th: $F = \overline{(\overline{x+y+z}) + (\overline{x+\overline{y}+\overline{z}}) + (\overline{\overline{x}+\overline{y}+z}) + (\overline{\overline{x}+\overline{y}+\overline{z}})}$

2)Findthe 1st and 2nd FC of functions:

$F1 = (a+b)(\overline{a}+\overline{b}+\overline{c})(a+c)$

$F2 = ab + abc + acd + bd$ (to do)

Boolean Algebra

Simplification of logical functions: Algebraic method

Any logical function represented either by written expression, by logic diagram or by truth table, can be simplified

Simplification consists of writing a function using a minimum number of terms and logic operators in order to save on circuits construction

The Algebraic simplification is realized by using the notions of Boolean algebra (fundamental laws, MORGAN's law, canonical forms)

Boolean Algebra

Simplification of logical functions: Algebraic method

Example 1: simplify the function F $= (a+b)(\overline{a}+\overline{b}+\overline{c})(a+c)$

$$F = (a+b)(\overline{a}+\overline{b}+\overline{c})(a+c) = (a\overline{b}+a\overline{c}+\overline{a}b+b\overline{c})(a+c)$$

$$F = a\overline{b}+a\overline{c}+ab\overline{c}+a\overline{b}c+\overline{a}bc = a\overline{b}+a\overline{c}+\overline{a}bc$$

$$F = a(\overline{b}+\overline{c})+\overline{a}bc = a\overline{bc}+\overline{a}bc = a \oplus bc$$

Example 2: simplifythe function F defined by the truth table

A	B	F
0	0	0
0	1	0
1	0	1
1	1	1

$F = (A\overline{B})+(AB) = A(\overline{B}+B) = A$ Using 1st CF

$F = (A+B)(A+\overline{B}) = A+(B\overline{B}) = A$ Using 2nd CF

Boolean Algebra

Simplification of logical functions: KARNAUGH table

The KARNAUGH TABLE is another form of the truth table, characterized by double entry table (Rows / Columns) for n variables the table must be as square as possible for p and q entries with p + q = n.

We have : $p=q=n/2$ where n is even and $|p-q|=1$ where n is odd

The table will hold 2^p lines and 2^q columns, we fill the variable entries according to the reflected binary code (Gray)

Example: for two variables p=1 and q=1

A	B	F
0	0	0
0	1	0
1	0	1
1	1	1

→

A \ B	0	1
0	0	0
1	1	1

Boolean Algebra

Simplification of logical functions: KARNAUGH table

Example: for three variables p=1 and q=2

x	y	z	F
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

x \ yz	00	01	11	10
0	0	1	0	1
1	1	1	0	0

Boolean Algebra

Simplification of logical functions: KARNAUGH table

Example: for four variables p=2 and q=2

ab \ cd	00	01	11	10
00	0	1	1	1
01	0	1	1	1
11	0	1	1	1
10	1	1	0	0

a	b	c	d	F
0	0	0	0	0
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	1
1	0	0	1	1
1	0	1	0	0
1	0	1	1	0
1	1	0	0	0
1	1	0	1	1
1	1	1	0	1
1	1	1	1	1

To obtain a canonical form of a function from the KARNAUGH table, we proceed in the same way as a standard truth table (Maxterms and Minterms)

Boolean Algebra

Simplification of logical functions: KARNAUGH table

Example: draw up the KARNAUGH table of the following function:

$$F1(A,B,C) = \overline{A}.B.C + A.\overline{B}.C + A.B.\overline{C} + A.B.C$$

0 1 1 1 0 1 1 1 0 1 1 1

SO :

A \ BC	00	01	11	10
0	0	0	1	0
1	0	1	1	1

$$F2(A,B,C) = (A+B+C)(A+B+\overline{C})(A+\overline{B}+C)(\overline{A}+B+C)$$

0 0 0 0 0 1 0 1 0 1 0 0

SO

A \ BC	00	01	11	10
0	0	0	1	0
1	0	1	1	1

Boolean Algebra

Simplification of logical functions: KARNAUGH table

To find the simplified form with the KARNAUGH table, we proceed as follows:

- Groups realization of 1, 2, 4,8, … adjacent terms (in general a power of 2)
- Minimization of groups (by maximizing terms in each group)
- Determining of simplified terms based on grouping 1 (minterms) or 0 (maxterms):
    - If the group have one value, then all variables of the term remain;
    - If the group have more values than we check and eliminate the variables that there states are changed on rows and columns, and we keep the others; respecting the variable value in relation to minterms or maxterm,
- The final logic expression is simplified form of the function

Boolean Algebra

Simplification of logical functions: Table of KARNAUGH

Example 1:

x \ yz	00	01	11	10
0	0	0	1	1
1	1	1	0	0

$$F = \overline{x}y + x\overline{y} = x \oplus y$$

OR

x \ yz	00	01	11	10
0	0	0	1	1
1	1	1	0	0

$$F = (x + y)(\overline{x} + \overline{y}) = x \oplus y$$

Boolean Algebra

Simplification of logical functions: Table of KARNAUGH

Example 2:

x \ yz	00	01	11	10
0	1	1	1	1
1	1	1	0	0

$$F = \overline{x} + \overline{y}$$

Example 3: using the symmetrical aspect of the KARNAUGH table

x \ yz	00	01	11	10
0	1	0	0	1
1	1	0	0	1

$$F = \overline{z}$$

Boolean Algebra

Simplification of logical functions: Table ofKARNAUGH

Example 4:

ab \ cd	00	01	11	10
00	0	1	1	1
01	0	1	1	1
11	0	1	1	1
10	1	1	0	0

$$F = (a + c + d)(\bar{b} + c + d)(\bar{a} + b + \bar{c})$$

Example 5:

ab \ cd	00	01	11	10
00	1	0	0	1
01	0	1	1	0
11	0	1	1	0
10	1	0	0	1

$F = \bar{b}\bar{d} + bd$ grouping of 1

Or $F = (b + \bar{d})(\bar{b} + d)$ grouping of 0

Exercise :Draw up the KARNAUGH table then simplify F:

$$F = (a + b)(ab + ac)(\bar{a}\bar{c} + b)$$