Chapter 4: Magnetostatics

1. Introduction

Magnetostatics studies magnetic fields generated by steady (time-independent) electric currents.

2. Magnetic Forces

2.1 Magnetic Force Acting on a Moving Charge

Consider a set of particles with charge $\boldsymbol{q_i}$, called *source charges*, and assume these charges are in motion.

Place at a point $\boldsymbol{P}$ a point-like test charge $\boldsymbol{q}$ at rest. This charge is subjected to a force proportional to $\boldsymbol{q}$, which we call the *electric force*, written as:

$$\vec{\boldsymbol{F}}_{\boldsymbol{e}} = \boldsymbol{q}\,\vec{\boldsymbol{E}} \tag{1}$$

$\vec{F}_e$: the electric force

$\vec{E}$: the electric field created by charges that are in motion, representing a generalization of the electrostatic field produced by charges that are essentially stationary. In general, the electric field depends on time.

Now suppose that the test charge placed at $\boldsymbol{P}$ has, at instant $\boldsymbol{t}$, a velocity $\vec{\boldsymbol{v}}$.

This charge is still subjected to the electric force $\vec{\boldsymbol{F}}_{\boldsymbol{e}} = \boldsymbol{q}\,\vec{\boldsymbol{E}}$ (the electric force does not depend on whether $\boldsymbol{q}$ is moving or stationary).

When the test charge $\boldsymbol{q}$ is in motion with velocity $\vec{\boldsymbol{v}}$, it is also subjected to an additional force called the *magnetic force*, denoted $\vec{\boldsymbol{F}}_{\boldsymbol{m}}$.

We associate with this magnetic force the magnetic induction field $\vec{\boldsymbol{B}}$. The relation between $\vec{\boldsymbol{F}}_{\boldsymbol{m}}$ and $\vec{\boldsymbol{B}}$ is:

$$\vec{F}_m = q\,\vec{v} \wedge \vec{B} \quad (2)$$

Thus, the total force $\vec{F}$ acting on a charge q in motion is:

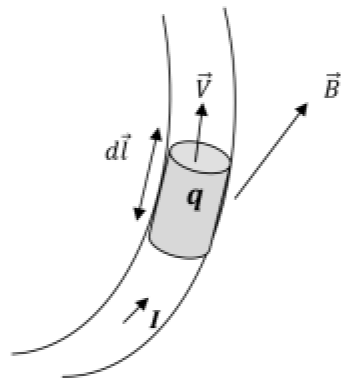
$$\vec{F} = q\,(\vec{E} + \vec{v} \wedge \vec{B}) \quad (3)$$

This is the **Lorentz force law**.

Notes

- If the test charge q is at rest, then $\vec{F}_m = \vec{0}$ because $\vec{v} = \vec{0}$. Therefore, $\vec{F} = \vec{F}_e = q\,\vec{E}$.
- If the source charges are at rest, no magnetic induction field is created: $\vec{B} = \vec{0}$. Thus $\vec{F}_m = \vec{0}$, and $\vec{F} = \vec{F}_e = q\,\vec{E}$.
- If $\vec{v}$ and $\vec{B}$ are collinear, then the cross product $\vec{v} \wedge \vec{B}$ is zero, giving $\vec{F} = \vec{F}_e = q\,\vec{E}$.
- If $q = 0$, then $\vec{F}_e = \vec{0}$, and $\vec{F}_m = \vec{0}$. The magnetic force acts only on a moving charge or on a conductor carrying a current.
- The direction of $\vec{F}_m$ is perpendicular to the plane defined by $\vec{v}$ and $\vec{B}$. For a positive charge, the vectors $\vec{v}$, $\vec{B}$, and $\vec{F}_m$ form a right-handed coordinate system (right-hand rule). For a negative charge, the force reverses direction.
- The right-hand rule or the corkscrew (right-hand screw) rule is used to determine the direction of $\vec{F}_m$.
- In direct-current circuits, the term $q\,\vec{E}$ is often negligible.
- In the International System of Units (**SI**), the unit of magnetic induction is the *tesla*, symbol **T**.

2.2 Magnetic Force Acting on a Conductor: Laplace's Law



Consider an element of length dl and cross-sectional area ds, carrying an electric current of intensity I. This current is composed of charged particles q moving with uniform velocity $\vec{v}$.

Let $\vec{B}$ be an external magnetic induction field, assumed to be uniform at every point of the element dl. Each particle within the element dl is subjected to a magnetic force: $\vec{F}_m$ $= q\,\vec{v} \wedge \vec{B}$

(since the particles of charge q are in motion). Therefore, the total magnetic force $d\vec{F}$ acting on the element dl is:

$$d\vec{F} = \sum (\, q\,\vec{v} \wedge \vec{B}) = (\sum q)\,(\,\vec{E} \wedge \vec{B})$$

where $\sum q$ is the total mobile charge contained in the considered element dl.

Let ρ_m be the mobile charge density. Then $\sum q = \rho_m\, ds\, dl$ so $d\vec{F} = (\rho_m\, ds\, dl\,)\,(\,\vec{E} \wedge \vec{B})$. With $\vec{J} = \rho_m\,\vec{v}$ the current density.

Let the current element vector $\vec{dl}$ be oriented in the same direction as the current (thus $\vec{dl}$ and $\vec{J}$ are parallel). We may therefore write: $d\vec{F} = ds\, J\ \vec{dl} \wedge \vec{B}$

Let $\boldsymbol{I}$ be the current intensity. Since $\boldsymbol{I = ds\,J}$, we can also write: $\boldsymbol{d\vec{F} = I\ \vec{dl} \wedge \vec{B}}$ This is **Laplace's law**.

Notes

- $\boldsymbol{d\vec{F}}$ is perpendicular to both $\boldsymbol{\vec{dl}}$ and $\boldsymbol{\vec{B}}$.
- The magnitude of $\boldsymbol{d\vec{F}}$ is $\boldsymbol{dF = I\,dl\,B\,sin\alpha}$
- The direction of $\boldsymbol{d\vec{F}}$ is given by the right-hand rule.

Summary

Every element of a filamentary circuit is subjected to an elementary Laplace force given by:

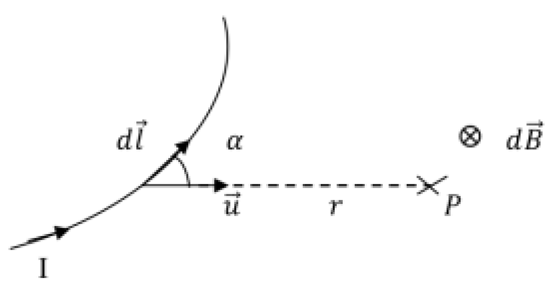
$$\boldsymbol{d\vec{F} = I\ \vec{dl} \wedge \vec{B}}\quad (\textbf{\textit{Laplace's law}})$$

The combined effect of all these forces is equivalent to a single resultant force:

$$\vec{F} = \int d\vec{F}.$$

3 Magnetic Induction Field Created by Currents: Biot–Savart Law

Consider a current element $\boldsymbol{\vec{dl}}$ originating at point $\boldsymbol{O}$ and carrying a current of intensity $\boldsymbol{I}$.



The magnetic field created by a current element is given by:

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{l}\wedge\vec{r}}{r^3} = \frac{\mu_0}{4\pi}\frac{Id\vec{l}\wedge\vec{u}}{r^2} \tag{4}$$

Biot–Savart’s law is an elemental relation that allows us to determine the elementary magnetic field $\vec{dB}$ produced by the current element $\vec{dl}$.

The current element $\vec{dl}$ is not a physically isolable entity; therefore, $\vec{dB}$ is not, strictly speaking, a physical law—rather, it is a **computational tool**.

The magnitude of $\vec{dB}$ is:

$$dB = \frac{\mu_0 I}{4\pi r^2} dl\sin(\alpha)$$

where α\alphaα is the angle between $\vec{u}$ and $\vec{dl}$.

Thus, the expression for the elementary field $\vec{dB}$ created by a current element is:

- In the case of a linear (filamentary) current distribution $I\vec{dl}$:

$$d\vec{B} = \frac{\mu_0}{4\pi}.\frac{Id\vec{l}\wedge P\vec{M}}{P\vec{M}^3}$$

- In the case of a surface current distribution $\vec{J}\,ds$:

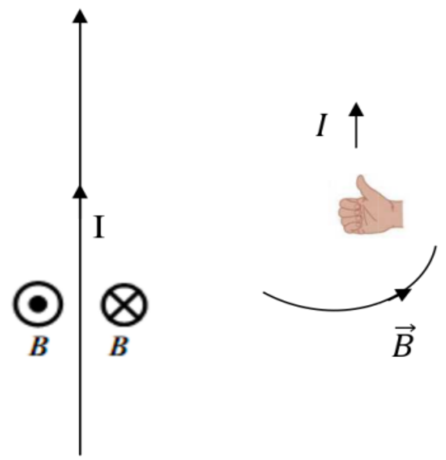
$$d\vec{B} = \frac{\mu_0}{4\pi}.\frac{\vec{J}ds\wedge P\vec{M}}{PM^3}$$

- In the case of a volume current distribution $\vec{J}d\tau$:

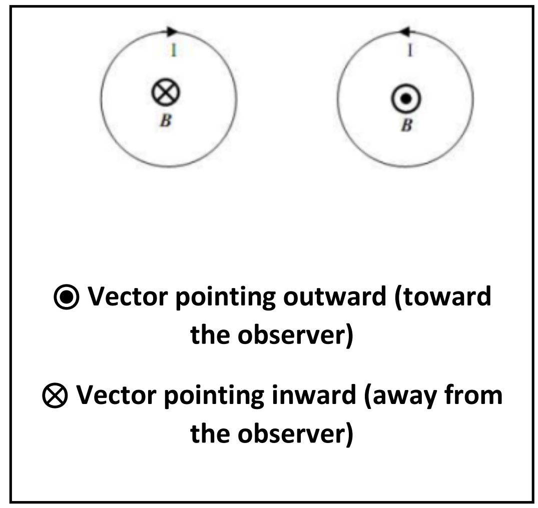
$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{\vec{J}d\tau\wedge P\vec{M}}{PM^3}$$

Direction of the Magnetic Field

Straight Wire: (Right-Hand Rule)



Loop: Right-Hand Screw Rule



4 Ampère's Theorem

The line integral of the magnetic field around a closed path is proportional to the current it encloses:

$$\oint_{\text{oriented contour}} \vec{B} \cdot \vec{dl} = \mu_0 \sum I_{\text{enclosed (algebraic)}} \tag{5}$$

- $\oint_{\text{oriented contour}}$: The oriented contour must be **closed** (it bounds a surface).
- $\vec{\boldsymbol{B}}$: the magnetic field created by all currents (whether enclosed or not).
- $\vec{\boldsymbol{dl}}$: defines the direction of traversal of the contour (arbitrary orientation).
- The chosen traversal direction defines a normal vector $\vec{\boldsymbol{n}}$ to the surface bounded by the contour (right-hand rule).
- If the contour follows a line of the magnetic field, then $\vec{\boldsymbol{B}}$ and $\vec{\boldsymbol{dl}}$ are collinear.
- $\boldsymbol{I}_{\textit{enclosed (algebraic)}}$: the algebraic sum of the currents linked with the contour.
  - Currents whose direction is consistent with $\vec{\boldsymbol{n}}$ are counted positively.

- Currents in the opposite direction are counted negatively.

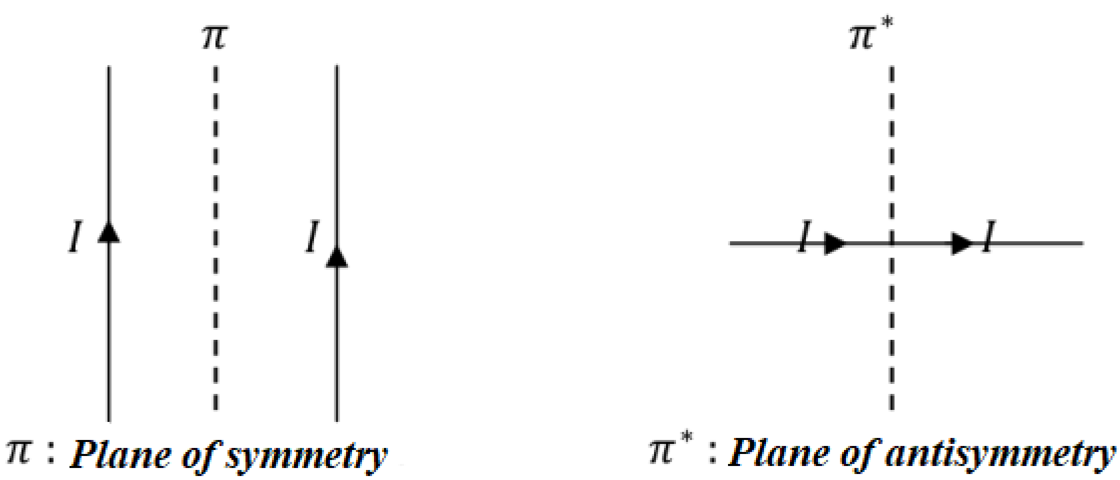
4.1 Computing the Magnetic Field $\vec{B}$ Using Ampère's Theorem

- Identify the invariances and symmetries of the current distribution.
- Choose a closed Ampèrian contour.
- Apply Ampère's theorem:

$$\oint_{\text{oriented contour}} \vec{B} \cdot d\vec{l} = \mu_0 \sum I_{\text{enclosed (algebraic)}} \tag{6}$$

4.1.1 Symmetry and Antisymmetry Planes of the Current Distribution

Definitions of symmetry and antisymmetry planes of currents:



- At any point lying on a symmetry plane of the currents, the magnetic field is perpendicular to that plane.
- At any point lying on an antisymmetry plane of the currents, the magnetic field lies within that plane.
- A symmetry plane of the currents is an antisymmetry plane of the magnetic field.
- An antisymmetry plane of the currents is a symmetry plane of the magnetic field.

- Magnetic field lines are closed; they circulate around the sources (the currents).

4.2 Integral Form of Ampère's Theorem

If volume currents are linked with the contour, the corresponding "enclosed" current is written:

$$I_{\text{enclosed (algebraic)}} = \iint \vec{J} \cdot d\vec{S}$$

Thus, the integral form of Ampère's theorem becomes:

$$\oint_{c\,\text{(oriented contour)}} \vec{B} \cdot d\vec{l} = \mu_0 \iint_S \vec{J} \cdot d\vec{S}$$

where ***S*** is any surface bounded by the oriented contour ***c***.

5 Magnetic Flux and Induction

5.1 Magnetic Flux

The magnetic flux through a surface ***S*** is given by:

$$\Phi_B = \int_S \vec{B} \cdot d\vec{S}$$

It represents the amount of magnetic field passing through a given surface.

5.2 Self-Inductance

Inductance is the property of a circuit that opposes variations in the current flowing through it. It is defined by:

$$\Phi_B = LI$$

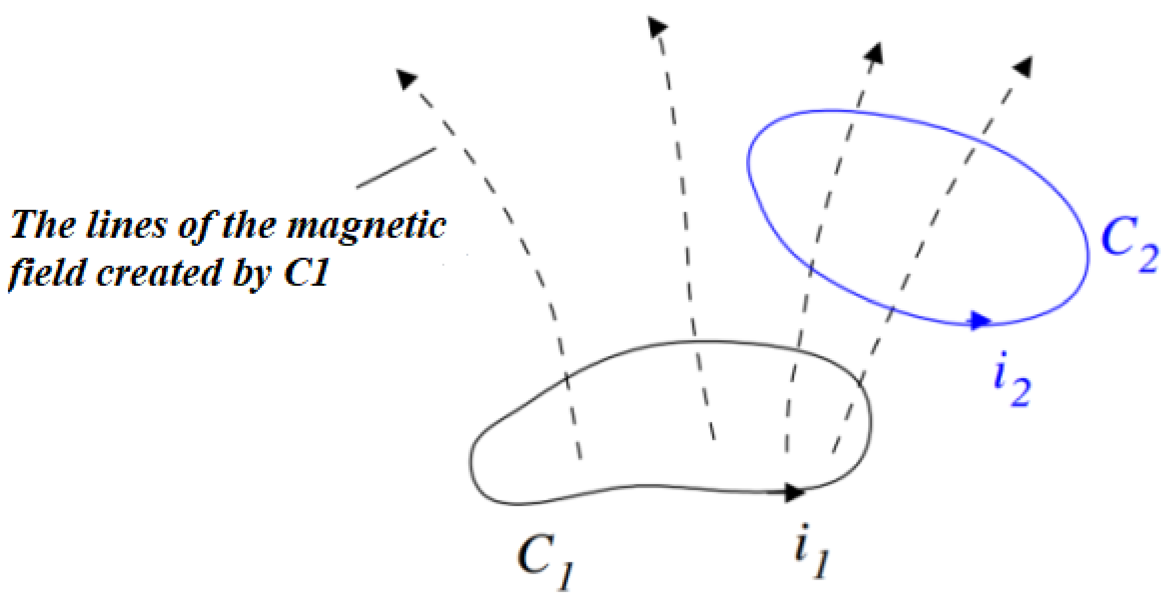
The coefficient ***L*** is called the self-inductance and depends on the geometry of the circuit.

5.3 Mutual Inductance

5.3.1 Definition

Mutual inductance is a coefficient that describes the influence of one magnetic circuit on another.

Consider two circuits C_1 carrying currents I_1 and I_2. The current I_1 produces in all space a magnetic field B_1. The flux generated by the magnetic field B_1 through C_2 is denoted Φ_{12}.



$$\Phi_{12} = \int\int_{S_2} \vec{B}_1.d\vec{S}_2 = \int\int_{S_2} \vec{\mathrm{rot}}\vec{A}_1.d\vec{S}_2 = \oint_{C_2} \vec{A}_1.d\vec{l}_2$$ (Stokes' theorem)

The expression for the vector potential generated by a filamentary circuit C_1 carrying a current I_1 is:

$$\vec{A}_1 = \frac{\mu_0}{4\pi} \oint_{C_1} I_1 \frac{d\vec{l}_1}{r_{12}}$$

The final expression for the flux is:

$$\Phi_{12} = I_1 . \left(\frac{\mu_0}{4\pi} \oint_{C_2} \oint_{C_1} \frac{d\vec{l}_1 d\vec{l}_2}{r_{12}} \right) = M_{12} I_1$$

so we write: $\phi_{12} = L_{12}$

where M_{12} is the mutual inductance between circuits C_1 and C_2.

$$M_{12} = \frac{\mu_0}{4\pi} \oint_{C_2} \oint_{C_1} \frac{d\vec{l}_1 d\vec{l}_2}{r_{12}}$$

We note that M_{12} is unchanged when indices 1 and 2 are swapped in the calculations; therefore:

$$\Phi_{21} = M_{21} I_2 = M_{12} I_2$$

6 Local Equations of Magnetostatics

6.1 Reminder on Differential Operators

Laplacian Operator

Definition:

It corresponds to the nabla operator applied twice to the considered function. It is most often applied to scalar fields, and the result is also a scalar field.

$$\Delta f = \vec{\nabla}.\vec{\nabla} f = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

The second partial derivative measures variations in slope ⇒ the scalar Laplacian represents the curvature of a scalar field.

The Laplacian links and combines the static description of a field (described by its gradient) with dynamic effects (the divergence).

The vector Laplacian is simply the scalar Laplacian applied to each component ($\boldsymbol{V_x(x, y, z)}$, $\boldsymbol{V_y(x, y, z)}$, $\boldsymbol{V_z(x, y, z)}$) of a vector field $\vec{\boldsymbol{V}}$**.**

$$\Delta \vec{V} = \vec{\nabla}.\vec{\nabla}\left(\vec{V}\right) = \begin{pmatrix} \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} \\ \frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_y}{\partial z^2} \\ \frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2} \end{pmatrix}$$

6.2 Local Equations of Magnetostatics

The local equations of magnetostatics are derived from Maxwell's equations under the assumption of a stationary magnetic field ($\boldsymbol{\partial \vec{B}/\partial t=0}$).

6.3 Gauss's Law for the Magnetic Field

The magnetic field has no scalar sources — meaning it cannot diverge from a point the same way an electric field diverges from an electric charge.
This implies that the divergence of the magnetic field is always zero:

$$\vec{\nabla} \cdot \vec{B} = 0$$

This means that magnetic field lines are always closed or extend to infinity.

Why?

- Comparison with the electric field: The electric field $\vec{\boldsymbol{E}}$ has sources and sinks: it starts from positive charges and ends on negative charges.

This is expressed by Maxwell–Gauss's equation:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

Here, $\boldsymbol{\rho}$ represents the charge density, meaning there are point sources of the electric field.

In the case of the magnetic field:

- Magnetic monopoles do not exist (as far as experimental physics has shown).
- Magnetic field lines are always closed: they form continuous loops.
- Gauss's law for $\vec{\boldsymbol{B}}$ is therefore:

$$\vec{\nabla} \cdot \vec{B} = 0$$

This means that there is no point where $\vec{\boldsymbol{B}}$ "appears" or "disappears," unlike $\vec{\boldsymbol{E}}$.

Concrete example:

- Around an infinite wire carrying a current, $\vec{\boldsymbol{B}}$ forms concentric circles: there is no beginning or end, just magnetic flux looping around.
- Around a magnet, the magnetic field $\vec{\boldsymbol{B}}$ comes out of the North pole and enters the South pole, but in reality, inside the magnet, it continues its loop.

Therefore, $\vec{\boldsymbol{B}}$ has no sources or sinks; it is always in a closed circuit.

6.4 Ampère's Law (local form)

Ampère's law states that the curl of the magnetic field is proportional to the current density:

$$\vec{\nabla} \wedge \vec{B} = \mu_0 \vec{J}$$

where $\vec{\boldsymbol{J}}$ is the electric current density.
This equation indicates that electric currents are the sources of magnetic fields.